The University of Melbourne

Semester 1 Assessment, 2003

Department of Mathematics and Statistics

620-221 Real and Complex Analysis

Instructions to Students:
All questions carry the same number of marks. All questions may be attempted but only marks from the best *ten* questions will be counted.

Identical Examination Papers: nil
Common content examinations: nil
Reading time: 15 minutes
Duration of examination: Three hours
Length of this question paper: 5 pages

Authorized materials:
Pens, rubbers, and rulers are authorized. No other materials are authorized; in particular, calculators are not authorised. Candidates are reminded that no written or printed material related to this subject may be brought into the examination. If you have any such material in your possession, you should immediately surrender it to an invigilator.

Instructions to Invigilators:
Script books only are required. Candidates are permitted to take this question paper with them at the end of the examination. No written or printed material related to the subject may be brought into the examination.

Reproduction of question paper: After the examination, this question paper may be reproduced and lodged in the Baillieu Library.

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1. (a) Find the argument of
$$\frac{(1+i)^3(1-i)^2}{(1+\sqrt{3}i)^2}.$$

   (b) If z_1 and z_2 are complex numbers, give a brief geometrical explanation for the following inequality:
$$|z_1| + |z_2| \geq |z_1 + z_2|.$$

   (c) Give a brief geometric description of the subset of the plane described by the following:
$$\{z : |z-1| < |z-i|\}.$$

2. (a) Explain carefully what is meant by saying that a subset of the complex plane is *open* and what is meant by saying that a subset is *closed*.

   (b) Give an example of a subset of the complex plane that is neither open nor closed.

   (c) Explain what is meant by a *domain*. Explain briefly why, if a single point is removed from a domain, then the result is still a domain.

3. Let S be the interior of the unit circle; that is, let $S = \{z : |z| < 1\}$. Show carefully that each point of the unit circle, that is $\{z : |z| = 1\}$, is a limit point of S. You should define what you mean by *limit point* and your proof should use that definition.

4. (a) State carefully the Heine-Borel Lemma (or theorem) concerning the covering of closed and bounded sets by open sets.

   (b) Suppose that $C_1, C_2, C_3, \ldots$ is a collection of non-empty closed subsets of the plane satisfying
$$C_1 \supseteq C_2 \supseteq C_3 \supseteq \ldots$$

and suppose that C_1 is bounded. Apply the Heine-Borel Lemma to the complements of the C_i to show that the intersection $\cap_{i=1}^{\infty} C_i$ is non-empty.

5. Find all real numbers a so that $u(x, y) = x^3 - axy^2$ is the real part of a function f of $z = x + iy$ which is analytic on the complex plane. Also find all such functions f.

6. Show carefully, stating any theorems used, that if a function is analytic in a domain and takes only real values, then it must be constant.

7. (a) Write down a power series expansion for the function $z \mapsto \dfrac{1}{1-z}$. Write down its radius of convergence.

   (b) Use the power series expansion above to find a power series expansion for $z \mapsto \dfrac{1}{(1-z)^3}$. What is the radius of convergence of this power series?

8. Calculate the radius of convergence of

   (a) $\displaystyle\sum_{n=0}^{\infty} \frac{n}{2^n} z^n$ (b) $\displaystyle\sum_{n=0}^{\infty} \cos(in) z^n$

9. Evaluate the integral

$$\int_{\gamma} \frac{\cos z}{z(z-2i)^2}\, dz$$

   where γ is

   (a) the circle $|z| = 1$; (b) the circle $|z - 2i| = 1$.

10. Find the Laurent expansion about 0 of

$$\frac{1}{z(z-1)(z-2)}$$

    on the annulus $\{z : 1 < |z| < 2\}$.

11. Calculate, using the residue theorem,

$$\int_{|z|=1} \frac{\exp(z)}{z^2(z^2-4)}\, dz.$$

12. Show the following using contour integration techniques. You should indicate where you believe that certain integrals tend to 0 but need not provide a proof.

$$\int_{-\infty}^{\infty} \frac{x^2+1}{x^4+1}\,dx = \pi\sqrt{2}.$$