The University of Melbourne

Semester 1 Assessment, 2003

Department of Mathematics and Statistics

620-221 Real and Complex Analysis

Instructions to Students:
All questions carry the same number of marks. All questions may be attempted but only marks from the best *ten* questions will be counted.

Identical Examination Papers: nil
Common content examinations: nil
Reading time: 15 minutes
Duration of examination: Three hours
Length of this question paper: 10 pages

Authorized materials:
Pens, rubbers, and rulers are authorized. No other materials are authorized; in particular, calculators are not authorised. Candidates are reminded that no written or printed material related to this subject may be brought into the examination. If you have any such material in your possession, you should immediately surrender it to an invigilator.

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1. (a) Find the argument of

$$\frac{(1+i)^3(1-i)^2}{(1+\sqrt{3}i)^2}.$$

Solution:
Let z denote the given complex number. Then

$$\begin{aligned} \arg z &= 3\times\arg(1+i)+2\times\arg(1-i)-2\times\arg(1+\sqrt{3}i) \\ &= 3\times\left(\frac{\pi}{4}\right)+2\times\left(-\frac{\pi}{4}\right)-2\times\left(\frac{\pi}{3}\right)=-\frac{\pi}{4}-\frac{2\pi}{3}=-\frac{5\pi}{1}2. \end{aligned}$$

Thus the argument is $5\pi/12$, which is also the prinipcal value

(b) If z_1 and z_2 are complex numbers, give a brief geometrical explanation for the following inequality:

$$|z_1|+|z_2|\geq|z_1+z_2|.$$

Solution:
The length of a side of a triangle is less than or equal to the sum of the lengths of the other two sides.

(c) Give a brief geometric description of the subset of the plane described by the following:

$$\{z : |z-1|<|z-i|\}.$$

Solution:
The half plane to the bottom right of the line through the origin which goes from bottom left to top right at an angle of $\pi/4$ to the axes.

2. (a) Explain carefully what is meant by saying that a subset of the complex plane is *open* and what is meant by saying that a subset is *closed.*

Solution:
A subset X of the plane is open if, for each point $x \in X$, there exists a disc $\mathcal{N}$ centered at x, of the form $\{z : |z - x| < r\}$ which also lies completely in X. A subset is closed exactly when its complement is open.

(b) Give an example of a subset of the complex plane that is neither open nor closed.
Solution:
The subset $\{z : 0 < \mathrm{Re}(z) \leq 1\}$ is neither closed nor open. The point 1 has no neighbourhood in the set and so the set is not open. The point 0 of the complement has no neighbourhood in the complement and so the set is not closed.

(c) Explain what is meant by a *domain*. Explain briefly why, if a single point is removed from a domain, then the result is still a domain.
Solution:
A domain is an open, path-connected set. If we omit only one point, then if x is another point in the domain then we can still find a neighbourhood containing x by ensuring that the radius is less than the distance from x to the omitted point. If x lies on a path between two points, we can vary the path very slightly to avoid x but stay in the domain.

3. Let S be the interior of the unit circle; that is, let $S = \{z : |z| < 1\}$. Show carefully that each point of the unit circle, that is $\{z : |z| = 1\}$, is a limit point of S. You should define what you mean by *limit point* and your proof should use that definition.
Solution:
A point x is a limit point of a set S if every neighbourhood of x contains a point of S other than x. Suppose that $|z| = 1$ and let $\mathcal{N}$ denote a neighbourhood of z. Suppose that $\mathcal{N}$ has radius ρ. Choose τ so that $0 < \tau < 1$ and $\tau > 1 - \rho$. We claim that τz is a point of $\mathcal{N}$ which lies in S. We have $|\tau z| = \tau|z| = \tau < 1$ and so $\tau z \in S$. Also $|z - \tau z| = |z|(1 - \tau) = 1 - \tau < \rho$ and so $\tau z \in \mathcal{N}$. Thus $\mathcal{N}$ contains the point τz of S, as required.

4. (a) State carefully the Heine-Borel Lemma (or theorem) concerning

the covering of closed and bounded sets by open sets.

Solution:

If a closed and bounded subset B of the plane is a subset of an arbitrary union $\cup_i O_i$ of open sets O_i then there is a finite subcollection $O_1, \dots O_n$ of these sets so that B is a subset of $O_1 \cup O_2 \cup \dots \cup O_n$.

(b) Suppose that $C_1, C_2, C_3, \dots$ is a collection of non-empty closed subsets of the plane satisfying

$$C_1 \supseteq C_2 \supseteq C_3 \supseteq \dots$$

and suppose that C_1 is bounded. Apply the Heine-Borel Lemma to the complements of the C_i to show that the intersection $\cap_{i=1}^{\infty} C_i$ is non-empty.

Solution:

Suppose that the intersection were empty. Then $\cap_i C_i = \emptyset$ and so $\cup_i(\mathbb{C} \setminus O_i) = \mathbb{C}$. Then $C_1 \subseteq \mathbb{C} = \cup_i(\mathbb{C} \setminus O_i)$ and so, by the Heine-Borel Lemma,

$$C_1 \subseteq (\mathbb{C} \setminus C_{i_1}) \cup \dots (\mathbb{C} \setminus C_{i_n}) = \mathbb{C} \setminus (C_{i_1} \cap \dots \cap C_{i_n})$$

for some $i_1, \dots, i_n$. But the C_i form a decreasing sequence of sets and so $(C_{i_1} \cap \dots \cap C_{i_n}) = C_{i_m}$ where i_m is the maximum of $i_1, \dots, i_n$. That is, $C_1 \subseteq \mathbb{C} \backslash C_{i_m}$ or, equivalently,the intersection of C_1 with C_{i_m} is empty. Since this intersection is C_{i_m} and assumed to be non-empty, this is not possible.

5. Find all real numbers a so that $u(x, y) = x^3 - axy^2$ is the real part of a function f of $z = x + iy$ which is analytic on the complex plane. Also find all such functions f.

Solution:

We use the fact that $u_{xx} + u_{yy} = 0$. We have $u_{xx} = 6x$ and $u_{yy} = -2ax$. Thus the only possible value of a is $a = 3$. If v is the imaginary part of the analytic function f, then $u_x = v_y$ and so $v_y = 3x^2 - 3y^2$. Thus $v = 3x^2 y - y^3 + g(x)$ for some function $g(x)$. We also have $v_x = -u_y = -6xy$ and $v_x = 6xy + g'(x)$. Thus $g'(x) = 0$ and $g(x)$ is a constant. Thus

$$f(z) = f(x + iy) = (x^3 - axy^2) + i(3xy^2 + y^3 + c) = z^3 + c$$

where z is a real constant.

6. Show carefully, stating any theorems used, that if a function is analytic in a domain and takes only real values, then it must be constant.
Solution:
We again use the Cauchy-Riemann equations. These state that, if a function is analytic in a domain and we can write it as $f(x+iy) = u(x,y) + iv(x,y)$ then $u_x = v_y$ and $u_y = -v_x$ throughout the domain. The assumptions imply that $v = 0$ and so $u_x = u_y = 0$. Thus $f'(z) = u_x + iv_x = 0$ throughout the domain. A theorem from the notes now states that f is constant throughout the domain.

7. (a) Write down a power series expansion for the function $z \mapsto \dfrac{1}{1-z}$. Write down its radius of convergence.
Solution:

$$\frac{1}{1-z} = \sum_{i=0}^{\infty} z^i.$$

The radius of convergence is 1.

(b) Use the power series expansion above to find a power series expansion for $z \mapsto \dfrac{1}{(1-z)^3}$. What is the radius of convergence of this power series?
Solution:
We know that a power series can be differentiated term-by-term within its disc of convergence. In this case, differentiating the function twice gives

$$\left(\frac{1}{1-z}\right)'' = \left(\frac{1}{(1-z)^2}\right)' = \frac{1}{(1-z)^3}.$$

and so, twice differentiating the series term-by-term, we get

$$\frac{1}{(1-z)^3} = \left(\sum_{i=1}^{\infty} iz^{i-1}\right)' = \sum_{i=2}^{\infty} i(i-1)z^{i-2} = \sum_{i=0}^{\infty}(i+1)(i+2)z^i.$$

The radius of convergence is the same as the original, that is 1.

8. Calculate the radius of convergence of

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{n}{2^n} z^n$ (b) $\displaystyle\sum_{n=0}^{\infty} \cos(in) z^n$

Solution:
(a) We use the ratio test:

$$\lim_{n\to\infty}\frac{\frac{n}{2^n}}{\frac{n+1}{2^{n+1}}}=\lim_{n\to\infty}\frac{n2^{n+1}}{(n+1)2^n}=\lim_{n\to\infty}2\frac{n}{(n+1)}=2.$$

Thus the radius of convergence is 2.

(b) We can either use the Cauchy-Hadamard formula:

$$\limsup_{n\to\infty}(\cos(in))^{1/n}=\limsup_{n\to\infty}(\frac{1}{2}(e^n+e^{-n})^{1/n}=e$$

and so the radius of convergence is $1/e$ or we can use the ratio test:

$$\begin{aligned}\lim_{n\to\infty}\frac{\cos(in)}{\cos(i(n+1))} &= \lim_{n\to\infty}\frac{e^{-n}+e^n}{e^{-(n+1)}+e^{n+1}}=\lim_{n\to\infty}\frac{e^n(1+e^{-2n})}{e^{n+1}(1+e^{-2(n+1)})}\\ &= \frac{1}{e}\frac{\lim_{n\to\infty}(1+e^{-2n})}{\lim_{n\to\infty}(1+e^{-2(n+1)})}=\frac{1}{e}.\end{aligned}$$

9. Evaluate the integral

$$\int_\gamma\frac{\cos z}{z(z-2i)^2}\,dz$$

where γ is

(a) the circle $|z|=1$; (b) the circle $|z-2i|=1$.

Solution:
In case (a), the function $g(z)=\cos(z)/(z-2i)^2$ is analytic in the disc $|z|\le 1$ and so we can apply the Cauchy Integral Formula to show that

$$\int_{|z|=1}\frac{\cos z}{z(z-2i)^2}\,dz=2\pi i g(0)=2\pi i\times\frac{1}{-4}=\frac{-\pi i}{2}.$$

In case (b), the function $h(z)=\cos(z)/z$ is analytic within the given circle and so we can apply the generalised Cauchy formula to show

$$\begin{aligned}\int_{|z-2i|=1} \frac{\cos z}{z(z-2i)^2}\,dz &= 2\pi i h'(2i) \\ &= 2\pi i \times \left(\frac{-\sin(z)\times z - \cos(z)\times 1}{z^2}\right)_{z=2i} \\ &= 2\pi i \times \frac{-2i\sin(2i)-\cos(2i)}{-4} \\ &= 2\pi i \times \frac{e^2-3e^{-2}}{-8} = \frac{-\pi i(e^2-3e^{-2})}{4}.\end{aligned}$$

10. Find the Laurent expansion about 0 of

$$\frac{1}{z(z-1)(z-2)}$$

on the annulus $\{z : 1 < |z| < 2\}$.
Solution:
We write the expression as

$$\begin{aligned}\frac{1}{z}\left(\frac{1}{(z-1)(z-2)}\right) &= \frac{1}{z}\left(\frac{1}{z-2}-\frac{1}{z-1}\right) \\ &= \frac{1}{z}\left(\frac{-1}{2(1-(z/2)}-\frac{1}{z(1-1/z)}\right) \\ &= \frac{1}{z}\left(\frac{-1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n - \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n\right) \\ &= \left(\frac{1}{z}\right)\sum_{n=-\infty}^{\infty} a_n z^n\end{aligned}$$

where $a_n = -1$ if n is negative and $a_n = -2^{-(n+1)}$ if $n \geq 0$. Thus, multiplying by $1/z$, the Laurent expansion is

$$\sum_{n=-\infty}^{\infty} b_n z^n$$

where $b_n = -1$ if $n \leq -2$ and $b_n = -2^{-(n+2)}$ if $n \geq -1$.

11. Calculate, using the residue theorem,

$$\int_{|z|=1} \frac{\exp(z)}{z^2(z^2-4)}\, dz.$$

Solution:
The integrand has simple poles at $z = \pm 2$ and a double pole at $z = 0$. Thus only the double pole is within the circle of integration. The residue of the integrand at $z = 0$ can be calculated as :

$$\begin{aligned}\operatorname{Res}\left[\frac{e^z}{z^2(z^2-4)}, 0\right] &= \lim_{z\to 0}\left\{\frac{d}{dz}\left(z^2 \times \frac{\exp(z)}{z^2(z^2-4)}\right)\right\} \\ &= \left.\frac{e^z(z^2-4) - e^z(2z)}{(z^2-4)^2}\right|_{z=0} = \frac{-4}{16} = -\frac{1}{4}.\end{aligned}$$

12. Show the following using contour integration techniques. You should indicate where you believe that certain integrals tend to 0 but need not provide a proof.

$$\int_{-\infty}^{\infty} \frac{x^2+1}{x^4+1}\, dx = \pi\sqrt{2}.$$

Solution:
We take a contour γ_R which is a semicircle of radius $R > 1$ centered at 0 and with diameter along the real axis. Then the function $(z^2+1)/(z^4+1)$ has simple poles at $z = \eta^k$ where $\eta = \exp(i\pi/8)$ and $k = 1, 3, 5, 7$. Of these, the first two will lie within the semi-disc bounded by γ_R. The residue at η is

$$\left.\frac{z^2+1}{4z^3}\right|_{\eta} = \frac{\eta^2+1}{4\eta^3} = \frac{\eta^3+\eta}{-4} = \frac{-\eta^{-1}+\eta}{-4} = -i\left(\frac{2\operatorname{Im}(\eta)}{4}\right).$$

Similarly the residue at $z = \eta^3$ is $-i(2\operatorname{Im}(\eta^3))/4$. But $\operatorname{Im}(\eta) = \operatorname{Im}(\eta^3) = \sqrt{2}/2$. Thus the sum of the two residues is

$$\frac{-2i}{4} \times 2\left(\frac{\sqrt{2}}{2}\right) = -\frac{i\sqrt{2}}{2}.$$

Thus, if $R > 1$,

$$\int_{\gamma_R} \frac{z^2+1}{z^4+1}\, dz = 2\pi i \times -\frac{i\sqrt{2}}{2} = \pi\sqrt{2}.$$

But, if we denote the curved part of γ_R by σ_R then

$$\int_{\gamma_R} \frac{z^2+1}{z^4+1}\,dz = \int_{\sigma_R} \frac{z^2+1}{z^4+1}\,dz + \int_{-R}^{R} \frac{x^2+1}{x^4+1}\,dx$$

and, as R goes to infinity, the integral along σ_R approaches 0. Thus

$$\int_{-\infty}^{\infty} \frac{x^2+1}{x^4+1}\,dx = \pi\sqrt{2}.$$