The University of Melbourne

Semester 1 Assessment, 2004

Department of Mathematics and Statistics

620-221 Real and Complex Analysis

Instructions to Students:
All questions carry the same number of marks. All questions may be attempted but only marks from the best *ten* questions will be counted.

Identical Examination Papers: nil
Common content examinations: nil
Reading time: 15 minutes
Duration of examination: Three hours
Length of this question paper: 5 pages

Authorized materials:
Pens, rubbers, and rulers are authorized. No other materials are authorized; in particular, calculators are not authorised. Candidates are reminded that no written or printed material related to this subject may be brought into the examination. If you have any such material in your possession, you should immediately surrender it to an invigilator.

Instructions to Invigilators:
Script books only are required. Candidates are permitted to take this question paper with them at the end of the examination. No written or printed material related to the subject may be brought into the examination.

Reproduction of question paper: After the examination, this question paper may be reproduced and lodged in the Baillieu Library.

All questions carry the same number of marks. All questions may be attempted but only marks from the best ten questions will be counted.

1. (a) Find the absolute value of

$$\frac{(1+3i)^4(4+3i)^2}{(3+i)^3(1-i)^2}.$$

   (b) Give a brief argument to show that, for all complex numbers z:

$$|z| \leq |\operatorname{Re}(z)| + |\operatorname{Im}(z)|$$

   where $\operatorname{Re}(z)$ and $\operatorname{Im}(z)$ denote the real and imaginary parts of z.

   (c) Sketch the subset of the plane described by the following:

$$\{z : |z| < 2 \text{ and } |z-2| > 1\}.$$

2. (a) Give careful definitions of an *open* subset of the complex plane and of a *domain*. Give an example of an open subset which is not a domain.

   (b) Sketch the image of the strip $S = \{z : -1 \leq \operatorname{Re}(z) \leq +1\}$ under the exponential map. (That is, sketch the region $\exp(S)$).

3. Define a *limit point* of a subset of the complex plane and state carefully the Bolzano-Weierstrass Theorem concerning limit points of bounded sets. Indicate briefly why the Theorem shows that a Cauchy convergent sequence has a limit.

4. Suppose that f is an entire function and we have, for $z = x + iy$ with x and y real,

$$f(z) = u(x) + iv(y)$$

   for some real valued functions u and v. Show that f must be of the form $f(z) = az + b$ for some constants a and b.

5. Calculate the first four terms (that is, up to the power of z^3) of the power series expansion, about 0, of the function

$$\exp\left(\frac{1}{1-z}\right).$$

6. Find a function f which is analytic on the complex plane excluding the non-positive part of the real axis (that is, on the 'cut plane' $\mathbb{C}\backslash(-\infty, 0])$ and which satisfies $f(x) = x^x$ for positive real x. Find $f(i)$ and show that $\overline{f(z)} = f(\overline{z})$.

7. (a) Find the disc of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{3^n(z-3)^n}{n^2}.$$

(b) If the function

$$\frac{1}{(z-1)(z-i)(z+2)}$$

is expanded in a power series about the point $z = -i$, what is the radius of convergence?

8. Evaluate the integral

$$\int_\gamma \frac{\exp(z^2)}{z^3(z-4)}\, dz$$

where γ is (a) the circle with centre 3 and radius 2 and (b) the circle with centre 0 and radius 1 (described in the usual anti-clockwise direction in both cases).

9. Find the Laurent expansion about $z = 1$ of the function

$$\frac{1}{(z-1)(z+2)}$$

valid for a domain which includes the point $z = 5$.

10. (a) Explain what is meant by an isolated singularity. Give examples of a removeable singularity, a pole and an essential singularity.

(b) Describe the singularities of

$$\frac{2\exp(z) - \exp(1/z)}{1-z}.$$

11. Calculate, using the residue theorem,

$$\int_C \frac{\exp(z)}{\sin^2(z)}\,dz$$

where C is the circle with centre 0 and radius 1 described in the usual anti-clockwise direction.

12. Calculate the following integral using contour integration techniques. (You should indicate where you believe that certain integrals tend to zero but need not provide a proof.)

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+3)(x^2+5)}.$$

 END OF PAPER