# The University of Melbourne

## Semester 1 Assessment, 2004

## Department of Mathematics and Statistics

## 620-221 Real and Complex Analysis

**Instructions to Students**:
All questions carry the same number of marks. All questions may be attempted but only marks from the best *ten* questions will be counted.

**Identical Examination Papers**: nil
**Common content examinations**: nil
**Reading time**: 15 minutes
**Duration of examination**: Three hours
**Length of this question paper**: 5 pages

**Authorized materials**:
Pens, rubbers, and rulers are authorized. No other materials are authorized; in particular, calculators are not authorised. Candidates are reminded that no written or printed material related to this subject may be brought into the examination. If you have any such material in your possession, you should immediately surrender it to an invigilator.

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*All questions carry the same number of marks. All questions may be attempted but only marks from the best ten questions will be counted.*

1. (a) Find the absolute value of

$$\frac{(1+3i)^4(4+3i)^2}{(3+i)^3(1-i)^2}.$$

**Solution:**

The absolute value is

$$\left|\frac{(1+3i)^4(4+3i)^2}{(3+i)^3(1-i)^2}\right| = \frac{|1+3i|^5|4+3i|^2}{|3+i|^5|1-i|^2} = \frac{\sqrt{10}^4 5^2}{\sqrt{10}^3\sqrt{2}^2} = \frac{\sqrt{10}\times 25}{2}.$$

(b) Give a brief argument to show that, for all complex numbers $z$:

$$|z| \leq |\operatorname{Re}(z)| + |\operatorname{Im}(z)|$$

where $\operatorname{Re}(z)$ and $\operatorname{Im}(z)$ denote the real and imaginary parts of $z$.

**Solution:**

Write $z = x + iy$ where $x = \operatorname{Re}(y)$ and $y = \operatorname{Im}(y)$ are real. Then, by the triangle inequality,

$$|z| = |x+iy| \leq |x|+|iy| = |x|+|i||y| = |x|+|y| = |\operatorname{Re}(z)|+|\operatorname{Im}(z)|.$$

(c) Sketch the subset of the plane described by the following:

$$\{z : |z| < 2 \text{ and } |z-2| > 1\}.$$

**Solution:**

2. (a) Give careful definitions of an *open* subset of the complex plane and of a *domain*. Give an example of an open subset which is not a domain.

**Solution:**

A subset $S$ of the complex plane is open if, given any point $z$ in $S$, there is a neighbourhood $N$ of $S$ (that is, a disc centered on $z$) so that $N \subseteq S$. A domain $D$ is an open subset of the plane which is path connected; that is, for $z$ and $w$ in $D$, there should exist a continuous function $f : [0,1] \to D$ so that $f(0) = z$ and $f(1) = w$. The union of $\{z : |z| < 1\}$ with $\{z : |z - 3| < 1\}$ is an open set which is not a domain since it is not path-connected.

(b) Sketch the image of the strip $S = \{z : -1 \leq \mathrm{Re}(z) \leq +1\}$ under the exponential map. (That is, sketch the region $\exp(S)$).

**Solution:**

The sketch should show an annulus with inner radius $e^{-1}$ and outer radius $e$.

3. Define a *limit point* of a subset of the complex plane and state carefully the Bolzano-Weierstrass Theorem concerning limit points of bounded sets. Indicate briefly why the Theorem shows that a Cauchy convergent sequence has a limit.

**Solution:**

A point $z$ is a limit point of a set $S$ if every neighbourhood of $z$ contains a point of $S$. The BW Theorem states that an infinite bounded set has a limit point.

Suppose that $\{z_n\}$ is a Cauchy convergent sequence. Then Cauchy convergence implies that the terms of the sequence become arbitrarily close and so, if it contains only finitely many different points then it must be eventually constant. In this case it must clearly have a limit. Fix $\epsilon$. Cauchy convergence implies that there exists $N$ so that, for $m, n \geq N$, $|z_m - z_n| < \epsilon$. Thus the tail of the sequence lies in an $\epsilon$-neighbourhood of $z_N$. Since there are only finitely many terms of the sequence not in the tail, it is clear the sequence is bounded. Thus, by the BW Theorem, the sequence has a limit point, $z$. Since every neighbourhood of $z$ contains some $z_n$, it is clear(?) that the sequence converges to $z$.

4. Suppose that $f$ is an entire function and we have, for $z = x + iy$ with $x$ and $y$ real,
$$f(z) = u(x) + iv(y)$$
for some real valued functions $u$ and $v$. Show that $f$ must be of the form $f(z) = az + b$ for some constants $a$ and $b$.

**Solution:**

The Cauchy-Riemann equations imply that $u'(x) = v'(y)$ for all $x$ and $y$. But then $u'(x)$ and $v'(y)$ must be constant, and equal to $a$ say. Thus $u'(x) = v'(y) = a$ and so

$$u(x) = ax + b \text{ and } v'(y) = ay + c.$$

Thus $f(z) = (ax + b) + i(ay + c) = a(x + iy) + (b + ic) = az + \beta$ with $a$ and $\beta$ constant.

5. Calculate the first four terms (that is, up to the power of $z^3$) of the power series expansion, about 0, of the function
$$\exp\left(\frac{1}{1-z}\right).$$

**Solution:**

We shall calculate this using $n!a_n = f^{(n)}(0)$. If the function is $f(z)$ then

$$f'(z) = \frac{1}{(1-z)^2}f(z) \quad f'' = \frac{2}{(1-z)^3}f(z) + \frac{1}{(1-z)^4}f(z)$$

$$f'''(z) = \frac{6}{(1-z)^4}f(z) + \frac{2}{(1-z)^5}f(z) + \frac{4}{(1-z)^5}f(z) + \frac{1}{(1-z)^6}f(z).$$

Thus

$$f(0) = e \quad f'(0) = e \quad f''(0) = 3e \quad f'''(0) = 13e.$$

Hence the series is

$$e + ez + \frac{3e}{2}z^2 + \frac{13e}{6}z^3 + \dots.$$

6. Find a function $f$ which is analytic on the complex plane excluding the non-positive part of the real axis (that is, on the 'cut plane' $\mathbb{C}\backslash(-\infty, 0]$) and which satisfies $f(x) = x^x$ for positive real $x$. Find $f(i)$ and show that $\overline{f(z)} = f(\overline{z})$.

**Solution:**

We claim $f(z) = \exp(z\,\mathrm{Log}(z))$, where Log is the branch of Log which is defined on this cut plane), satisfies the requirements. For it is defined and analytic on the cut plane and if $x$ is real and positive, then

$$f(x) = \exp(x\,\mathrm{Log}(x)) = \exp(x(\log(|x|)+i\,\mathrm{Arg}(x)) = \exp(x\log(x)) = x^x.$$

Observe that

$$\exp(\overline{u+iv}) = \exp(u-iv) = \exp(u)\overline{\exp(iv)} = \overline{\exp(u+iv)}.$$

Thus

$$i^i = \exp(i\,\mathrm{Log}(i)) = \exp(i(\log(|i|)+i\,\mathrm{Arg}(i))) = \exp(i(0+i\pi/2)) = \exp(-\pi/2).$$

Finally, if $z = x+iy$ then $\overline{z} = x-iy$ and $\mathrm{Arg}(\overline{z}) = -\mathrm{Arg}(z)$. Thus

$$\begin{aligned} f(\overline{z}) &= \exp(\overline{z}(\mathrm{Log}\,\overline{z})) = \exp(\overline{z}(\log(|\overline{z}| + i\,\mathrm{Arg}(\overline{z})))) \\ &= \exp(\overline{z}(\log(|z|) - i\,\mathrm{Arg}(z))) = \exp(\overline{z}\overline{\mathrm{Log}\,z}) = \exp(\overline{z\,\mathrm{Log}\,z}) \\ &= \overline{\exp(z\log z)} = \overline{f(z)}. \end{aligned}$$

7. (a) Find the disc of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{3^n(z-3)^n}{n^2}.$$

**Solution:**

We use the ratio test to establish the radius of convergence:

$$\lim_{n\to\infty} \frac{\frac{3^n}{n^2}}{\frac{3^{n+1}}{(n+1)^2}} = \frac{1}{3}\lim_{n\to\infty} \frac{(n+1)^2}{n^2} = \frac{1}{3}\lim_{n\to\infty}\left(1+\frac{2}{n}+\frac{1}{n^2}\right) = \frac{1}{3}.$$

Thus the disc of convergence is

$$\{z : |z-3| < 1/3\}.$$

(b) If the function

$$\frac{1}{(z-1)(z-i)(z+2)}$$

is expanded in a power series about the point $z = -i$, what is the radius of convergence?

**Solution:**

The radius of convergence is the distance from $-i$ to the nearest singularity of the function. That is $|-i-1| = \sqrt{2}$.

8. Evaluate the integral

$$\int_\gamma \frac{\exp(z^2)}{z^3(z-4)}\,dz$$

where $\gamma$ is (a) the circle with centre 3 and radius 2 and (b) the circle with centre 0 and radius 1 (described in the usual anti-clockwise direction in both cases).

**Solution:**

We can use Cauchy's (generalised) integral formula. (a) Write the integral as

$$\int_\gamma \frac{e^{z^2} z^{-3}}{(z-4)}\,dz$$

where $f(z) = e^{z^2} z^{-3}$ is analytic within the circle. Then the integral is $2\pi i f(0) = 2\pi i \frac{e^{16}}{64}$.

(b) Write the integral as

$$\int_\gamma \frac{e^{z^2}(z-4)^{-1}}{z^3}\,dz$$

where $g(z) = e^{z^2}(z-4)^{-1}$ is analytic within the circle. Then the integral is

$$\frac{2\pi i}{2!} g^{(2)}(0).$$

But

$$g'(z) = \frac{2ze^{z^2}}{z-4} - \frac{e^{z^2}}{(z-4)^2}$$

and

$$g''(z) = \frac{4z^2e^{z^2}}{z-4} + \frac{2e^{z^2}}{z-4} - \frac{2ze^{z^2}}{(z-4)^2} - \frac{2ze^{z^2}}{(z-4)^2} + \frac{2e^{z^2}}{(z-4)^3}.$$

Thus $g''(0) = \frac{2}{-4} + \frac{2}{(-4)^3}$ and so the integral is

$$\frac{2\pi i}{2!}\left(-\frac{1}{2} + \frac{2}{(-4)^3}\right) = \frac{2\pi i}{2!}\left(-\frac{17}{32}\right) = -\frac{17\pi i}{32}.$$

**or**

we can expand the integrand by power series

$$\begin{aligned}
\frac{\exp(z^2)}{z^3(z-4)} &= \frac{-1}{4z^3} \times \frac{\exp(z^2)}{1-z/4} \\
&= \frac{-1}{4z^3} \times \left(1 + z^2 + \frac{z^4}{2!} + \ldots\right) \times \left(1 + \frac{z}{4} + \left(\frac{z}{4}\right)^2 + \ldots\right) \\
&= \frac{-1}{4z^3} + \frac{-1}{16z^2} + \frac{-17}{64z} + \ldots.
\end{aligned}$$

Thus the values of the integral is $2\pi i \times (-17)/16 = -17\pi i/32$.

9. Find the Laurent expansion about $z = 1$ of the function
$$\frac{1}{(z-1)(z+2)}$$
valid for a domain which includes the point $z = 5$.

**Solution:**

Set $w = z - 1$, so that $z = 5$ becomes $w = 4$. We seek to expand $1/w(w+3)$ about $w = 0$ in a region containing $w = 4$. We thus use

$$\frac{1}{w(w+3)} = \frac{1}{w^2}\frac{1}{\left(1 - \frac{-3}{w}\right)} = \frac{1}{w^2}\sum_{j=0}^{\infty}\left(\frac{-3}{w}\right)^j = \sum_{j=-2}^{-\infty}(-3)^{-j-2}w^j$$

and the expansion is valid for $|-3/w| < 1$; that is, for $|w| > 3$. Returning to $z$, we have

$$\frac{1}{(z-1)(z+2)} = \sum_{j=-2}^{-\infty}(-3)^{-j-2}(z-1)^j$$

with an expansion valid for the region $|z-1| > 3$ which therefore includes $z = 5$.

10. (a) Explain what is meant by an isolated singularity. Give examples of a removeable singularity, a pole and an essential singularity.

**Solution:**

A point $z_0$ is an isolated singularity of a function $f$ if $f$ is not analytic at $z_0$ but there is a neighbourhood $N$ of $z_0$ so that $f$ is analytic in the deleted neighbourhood $N \setminus \{z_0\}$. Examples are $\sin z/z$ , $1/z$ ,$\exp(1/z)$, all at $z = 0$.

(b) Describe the singularities of

$$\frac{2\exp(z) - \exp(1/z)}{1 - z}.$$

**Solution:**

There are singularities at $z = 0$ and at $z = 1$. The former is an essential singularity since the Laurent expansion will involve infinitely many negative powers of $z$. The latter is a simple pole as $1 - z$ has a simple zero at $z = 1$ and $2\exp(z) - \exp(-z)$ is analytic and non-zero at $z = 1$.

11. Calculate, using the residue theorem,

$$\int_C \frac{\exp(z)}{\sin^2(z)}\, dz$$

where $C$ is the circle with centre 0 and radius 1 described in the usual anti-clockwise direction.

**Solution:**

There is a singularity at $z = 0$ which is a double pole as $\sin z$ has a simple zero at $z = 0$. We calculate the residue there as follows:

$$\begin{aligned}
\text{Res}[f, 0] &= \lim_{z\to 0}(f(z)z^2)' = \lim_{z\to 0}\left(\frac{\exp(z)}{(\sin(z)/z)^2}\right)' \\
&= \lim_{z\to 0}\left(\frac{\exp(z)}{(\sin(z)/z)^2} - \frac{\exp(z)(\sin z/z)'}{(\sin z/z)^2}\right) \\
&= 1 - \lim_{z\to 0}\left(\frac{\exp(z)(\cos z/z - \sin z/z^2)}{(\sin z/z)^2}\right) \\
&= 1 - \lim_{z\to 0}\left(\frac{z\cos z - \sin z}{z^2}\right) = 1 - 0 = 1.
\end{aligned}$$

Thus

$$\int_C \frac{\exp(z)}{\sin^2(z)}\,dz = 2\pi i \times 1 = 2\pi i.$$

**or**

$$\begin{aligned}
\frac{\exp(z)}{\sin^2 z} &= \frac{1 + z + \frac{z^2}{2!} + \dots}{\left(z - \frac{z^3}{3!} + \dots\right)^2} \\
&= \frac{1}{z^2} \times \left(1 + z + \frac{z^2}{2!} + \dots\right)\left(1 + 2\frac{z^2}{3!} + \dots\right) \\
&= \frac{1}{z^2} \times (1 + z + \dots).
\end{aligned}$$

Thus the residue of the function at $z = 0$ is 1 and the integral is calculated as above.

12. Calculate the following integral using contour integration techniques. (You should indicate where you believe that certain integrals tend to zero but need not provide a proof.)

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+3)(x^2+5)}.$$

**Solution:**

Set $f(z) = 1/(z^2+3)(z^2+5)$. This is defined everywhere in the complex plane except at $z = \pm\sqrt{3}i$ and at $z = \pm\sqrt{5}i$ where it has simple poles. Let $\Gamma_R$ be the semicircle on base $[-R, R]$ in the upper half-plane; assume that $R > 5$. Then $f$ has two singularities in the interior of $\Gamma_R$, at $z = \sqrt{3}i$ and at $z = \sqrt{5}i$. The residues there are

$$\operatorname{Res}[f(z), \sqrt{3}i] = \lim_{z\to\sqrt{3}i}(z-\sqrt{3}i)f(z) = \left.\frac{1}{(z+\sqrt{3}i)(z^2+5)}\right|_{z=\sqrt{3}i} = \frac{1}{4\sqrt{3}i}$$

and $\operatorname{Res}[f(z), \sqrt{5}i] = -1/4\sqrt{5}i$. Thus, by the residue theorem

$$\int_{\Gamma_R} \frac{dz}{(z^2+3)(z^2+5)} = 2\pi i\left(\frac{1}{4\sqrt{3}i} - \frac{1}{4\sqrt{5}i}\right) = \frac{2\pi}{4\sqrt{15}}(\sqrt{5}-\sqrt{3}) = \frac{\pi}{2\sqrt{15}}(\sqrt{5}-\sqrt{3}).$$

As $R$ approaches infinity, the part of the integral along the real axis approaches $\int_{-\infty}^{\infty} \frac{dx}{(x^2+3)(x^2+5)}$ and that along the curved portion approaches 0. Thus

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+3)(x^2+5)} = \frac{\pi}{2\sqrt{15}}(\sqrt{5}-\sqrt{3}).$$