The University of Melbourne

Semester 1 Assessment, 2005

Department of Mathematics and Statistics

620-221 Real and Complex Analysis

Instructions to Students:
All questions carry the same number of marks. All questions may be attempted but only marks from the best *ten* questions will be counted.

Identical Examination Papers: nil
Common content examinations: nil
Reading time: 15 minutes
Duration of examination: Three hours
Length of this question paper: 5 pages

Authorized materials:
Pens, rubbers, and rulers are authorized. No other materials are authorized; in particular, calculators are not authorised. Candidates are reminded that no written or printed material related to this subject may be brought into the examination. If you have any such material in your possession, you should immediately surrender it to an invigilator.

Instructions to Invigilators:
Script books only are required. Candidates are permitted to take this question paper with them at the end of the examination. No written or printed material related to the subject may be brought into the examination.

Reproduction of question paper: After the examination, this question paper may be reproduced and lodged in the Baillieu Library.

All questions carry the same number of marks. All questions may be attempted but only marks from the best ten questions will be counted.

1. (a) Find the argument of
$$\frac{(1+i)^3(1+\sqrt{3}i)^2}{i^3(1-i)^4}.$$

   (b) One of the following inequalities is true for all complex z and w. Indicate which one and give values of z and w for which the other is false.
$$(i)\ \ ||z|-|w|| \geq |z-w| \qquad (ii)\ \ ||z|-|w|| \leq |z-w|$$

   (c) Sketch the subset of the complex plane described by the following:
$$\{z : \operatorname{Re} z < 0 \text{ and } |z-i| < 1\}.$$

2. (a) Explain carefully what is meant by a limit point of a set S. Give an example, with explanation, of a set S with a limit point which is not in S.

   (b) If $z = x+iy$, what is the absolute value of $\exp(\exp(z))$ (or equivalently e^{e^z})?

3. (a) State carefully the Heine-Borel Theorem concerning the covering of closed and bounded sets of $\mathbb{C}$ by open sets.

   (b) Suppose that O is an open subset of the complex numbers with a closed and bounded subset $C \subseteq O$. Show that there are finitely many discs, with centre in C and which lie entirely in O, so that every point of C lies in one of these discs.

4. At which points of the complex plane is the function $f(z) = z|z|$ differentiable? At which points is it analytic?

5. Show that the Taylor series expansion, about $z = 2i$, of the function $\operatorname{Log} z$ is given by
$$\operatorname{Log} z = \log 2 + i\pi/2 - \frac{i(z-2i)}{2} + \frac{(z-2i)^2}{8} - \cdots - \frac{i^n(z-2i)^n}{n2^n} + \ldots.$$
   What is the radius of convergence of the series?

6. Calculate $\left(i^i\right)^i$ and $i^{(i^i)}$. In each case, decide whether the answer depends on the particular choices (that is, the choice of principal value) made when defining Arg or Log?

7. (a) Find the disk of convergence of the power series

$$\sum_{k=0}^{\infty} \frac{(z-1-i)^k}{1+2^k} .$$

(b) If the function

$$\frac{1}{(z-2)^2 \cos z}$$

is expanded in a power series about the point $z = 0$, what is the radius of convergence?

8. Evaluate the integral

$$\int_\gamma \frac{\cos(z^2)}{z^3(z-3)}\, dz$$

where γ is (a) the circle with centre 3 and radius 1 and (b) the circle with centre 0 and radius 2 (described in the usual anti-clockwise direction in both cases).

9. Find the Laurent expansion about $z = 0$ of the function

$$\frac{1}{(z-1)(z+3i)}$$

valid for a domain which includes the point $z = 2$. Describe the domain in which this Laurent series expansion is valid.

10. Suppose that f is an analytic function which has no zeroes on the real line. Let n be an integer. Show that the residue at $z = n$ of the function

$$g(z) = \pi f(z) \cot \pi z$$

is $f(n)$.

11. Calculate, using the residue theorem,

$$\int_C \frac{\cos\left(\exp(-z)\right)}{z^2(z-2)}\, dz$$

where C is the circle with centre 0 and radius 1 described in the usual anti-clockwise direction.

12. Calculate the following integral using contour integration techniques. (You should indicate where you believe that certain integrals tend to zero but need not provide a proof.)

$$\int_{-\infty}^{\infty} \frac{x^2\,dx}{(x^2+1)(x^2+4)}.$$