

The University of Melbourne

Semester 1 Assessment, 2005

Department of Mathematics and Statistics

620-221 Real and Complex Analysis

Instructions to Students:

All questions carry the same number of marks. All questions may be attempted but only marks from the best *ten* questions will be counted.

Identical Examination Papers: nil

Common content examinations: nil

Reading time: 15 minutes

Duration of examination: Three hours

Length of this question paper: 5 pages

Authorized materials:

Pens, rubbers, and rulers are authorized. No other materials are authorized; in particular, calculators are not authorised. Candidates are reminded that no written or printed material related to this subject may be brought into the examination. If you have any such material in your possession, you should immediately surrender it to an invigilator.

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All questions carry the same number of marks. All questions may be attempted but only marks from the best ten questions will be counted.

1. (a) Find the argument of

$$\frac{(1+i)^3(1+\sqrt{3}i)^2}{i^3(1-i)^4}.$$

Solution:

The argument is

$$\begin{aligned} \operatorname{Arg} \left\{ \frac{(1+i)^3(1+\sqrt{3}i)^2}{i^3(1-i)^4} \right\} &= 3 \operatorname{Arg}(1+i) + 2 \operatorname{Arg}(1+\sqrt{3}i) \\ &\quad - 3 \operatorname{Arg}(i) - 4 \operatorname{arg}(1-i) \\ &= \frac{3\pi}{4} + \frac{2\pi}{3} - \frac{3\pi}{2} - \frac{-4\pi}{4} \\ &= \frac{11\pi}{12}. \end{aligned}$$

- (b) One of the following inequalities is true for all complex z and w . Indicate which one and give values of z and w for which the other is false.

$$(i) \quad ||z| - |w|| \geq |z - w| \quad (ii) \quad ||z| - |w|| \leq |z - w|$$

Solution:

The second. For the first, take $z = 1, w = -1$. Then the LHS is 0 and the RHS is 2.

- (c) Sketch the subset of the complex plane described by the following:

$$\{z : \operatorname{Re} z < 0 \text{ and } |z - i| < 1\}.$$

Solution:

2. (a) Explain carefully what is meant by a limit point of a set S . Give an example, with explanation, of a set S with a limit point which is not in S .

Solution:

Bookwork. For the example, take S the open unit disk and any point on boundary.

- (b) If $z = x + iy$, what is the absolute value of $\exp(\exp(z))$ (or equivalently e^{e^z})?

Solution:

$\exp(x + iy) = \exp(x) \exp(iy) = \exp(x) \cos y + i \exp(x) \sin y$ and so

$$\begin{aligned} \exp(\exp(z)) &= \exp(\exp(x) \cos y + i \exp(x) \sin y) \\ &= \exp(\exp(x) \cos y) \times \exp(i \exp(x) \sin y). \end{aligned}$$

Since the latter term in this product is of the form $\exp(it)$ with t real, it has absolute value 1. Thus

$$|\exp(\exp(z))| = \exp(\exp(x) \cos y).$$

3. (a) State carefully the Heine-Borel Theorem concerning the covering of closed and bounded sets of \mathbb{C} by open sets.

Solution:

Bookwork

- (b) Suppose that O is an open subset of the complex numbers with a closed and bounded subset $C \subseteq O$. Show that there are finitely many discs, with centre in C and which lie entirely in O , so that every point of C lies in one of these discs.

Solution:

As O is open and $C \subseteq O$, for each point x of C there is a neighbourhood N_x of x lying in O . Thus C is covered by the collection of all neighbourhoods N_x with $x \in C$. As C is closed and bounded, the Heine-Borel theorem implies that a finite collection of these neighbourhoods covers C . This finite collection of neighbourhoods (which are disks) answers the problem.

4. At which points of the complex plane is the function $f(z) = z|z|$ differentiable? At which points is it analytic?

Solution:

Note that if $z = x + iy$ then $f(z) = x(x^2 + y^2)^{\frac{1}{2}} + iy(x^2 + y^2)^{\frac{1}{2}}$. Set

$$u(x, y) = x(x^2 + y^2)^{\frac{1}{2}} \quad v(x, y) = y(x^2 + y^2)^{\frac{1}{2}}.$$

Then

$$u_x = (x^2 + y^2)^{\frac{1}{2}} + x \times \frac{1}{2} \times 2x \times (x^2 + y^2)^{-\frac{1}{2}} = \frac{2x^2 + y^2}{(x^2 + y^2)^{\frac{1}{2}}}.$$

Similarly,

$$u_y = \frac{xy}{(x^2 + y^2)^{\frac{1}{2}}} \quad v_x = \frac{xy}{(x^2 + y^2)^{\frac{1}{2}}} \quad v_y = \frac{x^2 + 2y^2}{(x^2 + y^2)^{\frac{1}{2}}}.$$

Applying the Cauchy-Riemann equations, $u_x = v_y$ implies that $x^2 = y^2$ and $u_y = -v_x$ implies that $xy = 0$. Thus, if the functions are to be differentiable at z then $x = -y = 0$ and so $z = 0$. Conversely,

$$\lim_{z \rightarrow 0} \frac{z|z|}{z} = \lim_{z \rightarrow 0} |z| = 0$$

and so the function is differentiable at 0. As the function is differentiable at only one point of the plane, it is analytic nowhere.

5. Show that the Taylor series expansion, about $z = 2i$, of the function $\text{Log } z$ is given by

$$\text{Log } z = \log 2 + i\pi/2 - \frac{i(z - 2i)}{2} + \frac{(z - 2i)^2}{8} - \dots - \frac{i^n(z - 2i)^n}{n2^n} + \dots$$

What is the radius of convergence of the series?

Solution:

We shall calculate this using $n!a_n = f^{(n)}(2i)$. If $f(z) = \text{Log } z$ then we have $f'(z) = 1/z$ and $f''(z) = -1/z^2$ and, in general, $f^{(n)}(z) = (-1)^{n-1}(n-1)!/z^n$. Thus, if $z = 2i$

$$a_n = (-1)^{n-1} \frac{(n-1)!}{n!} \frac{1}{(2i)^n} = (-1)^{n-1} \frac{1}{n} \frac{(-i)^n}{2^n} = -\frac{1}{n2^n} i^n.$$

Note that $\text{Log}(2i) = \log|2i| + i \text{Arg}(2i) = \log 2 + i\pi/2$. Thus

$$\begin{aligned} \text{Log } z &= \log 2 + i\pi/4 - \frac{i(z-2i)}{2} + \frac{(z-2i)^2}{8} + \frac{i(z-2i)^3}{24} + \\ &\quad \dots + \frac{-i^n(z-2i)^n}{n2^n} + \dots \end{aligned}$$

The radius of convergence is the distance from $2i$ to the nearest singularity of \log , which is at $z = 0$. Thus the radius of convergence is 2.

6. Calculate $(i^i)^i$ and $i^{(i^i)}$. In each case, decide whether the answer depends on the particular choices (that is, the choice of principal value) made when defining Arg or Log ?

Solution:

$(i^i)^i = \exp(i \text{Log}(i^i))$ and

$$\begin{aligned} \text{Log}(i^i) &= \text{Log}(\exp(i \text{Log } i)) = \text{Log}(\exp(i(\pi/2 + 2in\pi))) \\ &= \text{Log}(\exp(-\pi/2 - 2n\pi)) = -\pi/2 - 2n\pi \end{aligned}$$

where n is an arbitrary integer. Thus

$$(i^i)^i = \exp(i(-\pi/2 - 2n\pi)) = -i$$

and the answer does not depend on the choice of principal value in this case. (Alternatively, $(i^i)^i = i^{i \times i} = i^{-1} = -i$.)

On the other hand, $i^{(i^i)} = \exp(-\pi/2)$ as above, and so

$$\begin{aligned} i^{(i^i)} &= \exp(i^i \log(i)) = \exp(\exp(-\pi/2) \times i(\pi/2 + 2n\pi)) \\ &= \exp(i\pi/2 \exp(-\pi/2)) \times \exp(2in\pi \exp(-\pi/2)) \\ &= e^{i\pi/2 e^{-\pi/2}} e^{2in\pi e^{-\pi/2}}. \end{aligned}$$

The first expression does not depend on the choice of principal value; the second one appears to.

7. (a) Find the disk of convergence of the power series

$$\sum_{k=0}^{\infty} \frac{(z-1-i)^k}{1+2^k}.$$

Solution:

We use the ratio test to establish the radius of convergence:

$$\lim_{k \rightarrow \infty} \frac{\frac{1}{1+2^k}}{\frac{1}{1+2^{k+1}}} = \lim_{k \rightarrow \infty} \frac{1+2^{k+1}}{1+2^k} = \lim_{k \rightarrow \infty} \frac{\frac{1}{2^k} + 2}{\frac{1}{2^k} + 1} = 2.$$

Thus the disc of convergence is

$$\{z : |z - 1 - i| < 2\}.$$

(b) If the function

$$\frac{1}{(z-2)^2 \cos z}$$

is expanded in a power series about the point $z = 0$, what is the radius of convergence?

Solution:

The radius of convergence is the distance from 0 to the nearest singularity of the function. The singularities occur at $z = 2$ and all points $z = \pi/2 + n\pi$. The nearest singularity is therefore at $z = \pi/2$ and the radius of convergence is $\pi/2$.

8. Evaluate the integral

$$\int_{\gamma} \frac{\cos(z^2)}{z^3(z-3)} dz$$

where γ is (a) the circle with centre 3 and radius 1 and (b) the circle with centre 0 and radius 2 (described in the usual anti-clockwise direction in both cases).

Solution:

We can use Cauchy's (generalised) integral formula. (a) Write the integral as

$$\int_{\gamma} \frac{\cos(z^2)/(z^3)}{(z-3)} dz$$

where $f(z) = \cos(z^2)/z^3$ is analytic within the circle. Then the integral is $2\pi i f(3) = 2\pi i(\cos 9)/27$.

(b) Write the integral as

$$\int_{\gamma} \frac{\cos z^2/(z-3)}{z^3} dz$$

where $g(z) = \cos z^2/(z - 3)$ is analytic within the circle. Then the integral is

$$\frac{2\pi i}{2!}g^{(2)}(0).$$

But

$$g'(z) = \frac{-2z \sin(z^2)(z - 3) - \cos(z^2)}{(z - 3)^2}$$

and

$$\begin{aligned} g''(z) &= \frac{1}{(z - 3)^4} \times \\ &\times [(-4z + 6) \sin(z^2) - (2z^2 + 6z)(2z) \cos(z^2) + 2z \sin(z^2)](z - 3)^2 \\ &- 2(z - 3)[-2z \sin(z^2)(z - 3) - \cos(z^2)] \end{aligned}$$

Thus $g''(0) = \frac{[0] - (-6)[-1]}{(-3)^4} = \frac{-6}{81} = -\frac{2}{27}$ and so the integral is

$$\frac{2\pi i}{2!} \times \frac{-2}{27} = -\frac{2\pi i}{27}.$$

Alternatively, we can write $g(z)$ as

$$\begin{aligned} g(z) &= -\frac{\cos(z^2)}{3(1 - \frac{z}{3})} = \frac{-1}{3} \left(1 - \frac{z^4}{2!} + \dots\right) \left(1 + \frac{z}{3} + \frac{z^2}{9} + \dots\right) \\ &= \frac{-1}{3} - \frac{z}{9} - \frac{z^2}{27} - \dots \end{aligned}$$

Thus

$$\frac{g(z)}{z^3} = \frac{-1}{3z^3} - \frac{1}{9z^2} - \frac{1}{27z} - \dots$$

and so the value of the integral is $2\pi i \times -1/27$, as before.

9. Find the Laurent expansion about $z = 0$ of the function

$$\frac{1}{(z - 1)(z + 3i)}$$

valid for a domain which includes the point $z = 2$. Describe the domain in which this Laurent series expansion is valid.

Solution:

Write

$$\begin{aligned} \frac{1}{(z-1)(z+3i)} &= \frac{1}{1+3i} \left(\frac{1}{z-1} - \frac{1}{z+3i} \right) = \\ &= \frac{1}{1+3i} \left(\frac{1}{z(1-\frac{1}{z})} - \frac{1}{3i(1-\frac{-z}{3i})} \right) \end{aligned}$$

noting that, when $z = 2$, $|1/z| < 1$ and $|-z/3i| < 1$. We can expand the two terms above using geometric series to obtain

$$\frac{1}{1-\left(\frac{1}{z}\right)} = \sum_{k=0}^{\infty} z^k$$

and

$$\frac{1}{3i\left(1-\left(\frac{-z}{3i}\right)\right)} = \frac{1}{3i} \sum_{k=0}^{\infty} \left(\frac{-z}{3i}\right)^k.$$

Thus the Laurent series is $\sum_{k=-\infty}^{\infty} a_k z^k$ where

$$a_i = \begin{cases} \frac{1}{1+3i} & \text{if } k < 0 \\ \frac{(-1)^k}{(1+3i)(3i)^{k+1}} & \text{if } k \geq 0. \end{cases}$$

The region of convergence is the annulus $\{z : 1 < |z| < 3\}$.

10. Suppose that f is an analytic function which has no zeroes on the real line. Let n be an integer. Show that the residue at $z = n$ of the function

$$g(z) = \pi f(z) \cot \pi z$$

is $f(n)$.

Solution:

Change variables by $w = z - n$ so we are calculating the residue at $w = 0$. Then

$$g(z) = g(w+n) = \pi f(w+n) \frac{\cos(\pi(w+n))}{\sin(\pi(w+n))}.$$

But $\cos(\pi(w+n)) = \cos(\pi w) \cos(\pi n) - \sin(\pi w) \sin(\pi n) = \cos(\pi w)$ and $\sin(\pi(w+n)) = \sin(\pi w) \cos(\pi n) + \cos(\pi w) \sin(\pi n) = \sin(\pi w)$. Thus

$$g(z) = \pi f(w+n) \frac{\cos(\pi(w))}{\sin(\pi(w))}$$

and g has a simple pole at $w = 0$ (as \sin has a simple zero there). Thus the residue of g at $w = 0$ is

$$\lim_{w \rightarrow 0} w \times \pi f(w+n) \frac{\cos(\pi(w))}{\sin(\pi(w))} = \lim_{w \rightarrow 0} \frac{f(w+n) \cos(\pi w)}{\sin(\pi w)/(\pi w)} = f(n) \times \frac{1}{1} = f(n)$$

as required.

11. Calculate, using the residue theorem,

$$\int_C \frac{\cos(\exp(-z))}{z^2(z-2)} dz$$

where C is the circle with centre 0 and radius 1 described in the usual anti-clockwise direction.

Solution:

There is a singularity at $z = 0$ which is a double pole. We calculate the residue there as follows:

$$\begin{aligned} \text{Res}[f, 0] &= \lim_{z \rightarrow 0} (f(z)z^2)' = \lim_{z \rightarrow 0} \left(\frac{\cos(\exp(-z))}{(z-2)} \right)' \\ &= \lim_{z \rightarrow 0} \left(\frac{\exp(-z) \sin(\exp(-z))(z-2) - \cos(\exp(-z))}{(z-2)^2} \right) \\ &= \frac{1 \times \sin(1) \times (-2) - \cos(1)}{4} \\ &= -\frac{2 \sin 1 + \cos 1}{4}. \end{aligned}$$

Thus

$$\int_C \frac{\cos(\exp(-z))}{z^2(z-2)} dz = 2\pi i \times -\frac{2 \sin 1 + \cos 1}{4} = -\pi i \times \frac{2 \sin 1 + \cos 1}{2}.$$

12. Calculate the following integral using contour integration techniques. (You should indicate where you believe that certain integrals tend to zero but need not provide a proof.)

$$\int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2+1)(x^2+4)}.$$

Solution:

Set $f(z) = z^2/(z^2 + 1)(z^2 + 4)$. This is defined everywhere in the complex plane except at $z = \pm i$ and at $z = \pm 2i$ where it has simple poles. Let Γ_R be the semicircle on base $[-R, R]$ in the upper half-plane; assume that $R > 3$. Then f has two singularities in the interior of Γ_R , at $z = i$ and at $z = 2i$. The residues there are

$$\operatorname{Res}[f(z), i] = \lim_{z \rightarrow i} (z - i)f(z) = \frac{z^2}{(z + i)(z^2 + 4)} \Big|_{z=i} = \frac{-1}{6i}$$

and

$$\operatorname{Res}[f(z), 2i] = \lim_{z \rightarrow 2i} (z - 2i)f(z) = \frac{z^2}{(z + 2i)(z^2 + 1)} \Big|_{z=2i} = \frac{-4}{-12i} = \frac{1}{3i}.$$

Thus, by the residue theorem,

$$\int_{\Gamma_R} \frac{dz}{(z^2 + 1)(z^2 + 4)} = 2\pi i \left(\frac{-1}{6i} + \frac{1}{3i} \right) = \frac{2\pi i}{6i} = \frac{\pi}{3}.$$

As R approaches infinity, the part of the integral along the real axis approaches $\int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2+1)(x^2+4)}$ and that along the curved portion approaches 0. Thus

$$\int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2 + 1)(x^2 + 4)} = \frac{\pi}{3}.$$