The University of Melbourne

Semester 1 Assessment, 2006

Department of Mathematics and Statistics

620-221 Real and Complex Analysis

Instructions to Students:
All questions carry the same number of marks.

Identical Examination Papers: nil
Common content examinations: nil
Reading time: 15 minutes
Duration of examination: Three hours
Length of this question paper: 5 pages

Authorised materials:
Pens, rubbers, and rulers are authorised. No other materials are authorised; in particular, calculators are not authorised. Candidates are reminded that no written or printed material related to this subject may be brought into the examination. If you have any such material in your possession, you should immediately surrender it to an invigilator.

Instructions to Invigilators:
Script books only are required. Candidates are permitted to take this question paper with them at the end of the examination. No written or printed material related to the subject may be brought into the examination.

All questions carry the same number of marks.

1. (a) Find the absolute value of

$$\frac{(2+2i)^7}{i^{29}(1-i)^{21}}.$$

(b) If the real part of z is positive, what can we say about $\exp(z)$?

(c) Describe the set of points z on the complex plane satisfying $|z-1| = \sqrt{2}|z|$.

2. (a) Explain carefully what is meant by a *limit point* of a subset of the plane. A set S is a non-empty subset of the plane. The set of all the limit points of S is denoted by T. Show that if T has limit points then these limit points lie in T.

(b) If $\{z_n : n = 1, 2, 3, \ldots\}$ is a sequence of complex numbers converging to a limit z (that is $\lim_{n\to\infty} z_n = z$) and if $z \neq 0$, then show that there is a natural number N so that $n > N$ implies that $z_n \neq 0$.

3. Let $C_1 \supseteq C_2 \supseteq C_3 \supseteq \ldots$ be a sequence of non-empty subsets of the plane with each contained in the previous one. Suppose that the C_i are closed, bounded and non-empty. Show that the C_i have a point in common. You may, if you wish, use the following steps.

Let O_i be the complement, in the plane, of C_i. If the intersection of the C_i is empty show that a finite union of the O_i must contain C_1. Explain why this is impossible and so complete the proof.

4. (a) Find a complex number w so that $\cos w = 2$.

(b) Show that

$$|\sin z|^2 + |\cos z|^2 = 1$$

only when z is real.

5. Suppose that f is an entire function with

$$f(z) = u(x, y) + iv(x, y)$$

where $z = x + iy$ and u, v are real valued functions of x and y.

If $u(x, y) = v(x, y)^2$ for all x and y, show, using the Cauchy Riemann equations, that f is a constant function.

6. The function f given by $f(z) = z^z$ is defined by using the principal value of the logarithm. At which points of the plane is $f(z)$ analytic? By calculating derivatives, or otherwise, find the first three terms in the power series, centered at 1, for $f(z)$.

7. Calculate the radius of convergence of

   (a) $\displaystyle\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} z^n$ (b) $\displaystyle\sum_{n=0}^{\infty} \log n \, z^n$.

8. Evaluate the integral
$$\int_C \frac{\cos z \, dz}{z(z + 3i)^2}$$
   where (a) C is the circle with centre 0 and radius 2, and (b) C is the circle with centre $-3i$ and radius 1.

9. Find the Laurent expansion about $z = 1$ (that is, using powers of $(z - 1)$) of the function
$$\frac{1}{z(z - i)(z - 1)}$$
   valid for a domain which includes the point $z = -2$. Describe the domain in which this Laurent series expansion is valid.

10. Let f be an entire function and suppose that, for some real positive constant M,
$$|f(z)| \leq M|z|$$
    for all z.

    (a) Set $g(z) = f(z)/z$, for $z \neq 0$. Show that g has a removeable singularity at $z = 0$.

    (b) Deduce that $g(z)$ is a constant function (for $z \neq 0$) and so that $f(z) = Kz$ for some constant K and all z.

11. Calculate, using the residue theorem,

$$\int_C \frac{\sin \pi z}{z^4 - 1}\, dz$$

where C is the circle with centre 0 and radius 2 described in the usual anti-clockwise direction.

12. Show the following integral using contour integration techniques. (You should indicate where you believe that certain integrals tend to zero but need not provide a proof.)

$$\int_{-\infty}^{\infty} \frac{dx}{(x^6+1)} = \frac{2\pi}{3}$$