The University of Melbourne

Semester 1 Assessment, 2007

Department of Mathematics and Statistics

620-221 Real and Complex Analysis

Instructions to Students:
All questions carry the same number of marks.

Identical Examination Papers: nil
Common content examinations: nil
Reading time: 15 minutes
Duration of examination: Three hours
Length of this question paper: 5 pages

Authorised materials:
Pens, rubbers, and rulers are authorised. No other materials are authorised; in particular, calculators are not authorised. Candidates are reminded that no written or printed material related to this subject may be brought into the examination. If you have any such material in your possession, you should immediately surrender it to an invigilator.

Instructions to Invigilators:
Script books only are required. Candidates are permitted to take this question paper with them at the end of the examination. No written or printed material related to the subject may be brought into the examination.

All questions carry the same number of marks.

1. (a) Find the argument of
$$\frac{(1+i)^7}{i^{19}(1-i)^{12}}.$$

   (b) Give a geometrical interpretation of the inequality
$$|z+w| \leq |z| + |w|$$
where z and w are complex numbers.

   (c) Describe the set of points z on the complex plane satisfying $|z| < |z-2-2i|$.

2. (a) Explain carefully what is meant by an *open* subset of the plane. Give an example, with brief justification, of a subset of the plane which is neither open nor closed.

   (b) Is the set consisting of a single point open or closed or neither? Give brief reasons.

   (c) Give a precise definition of a *path* in the complex plane. Using your definition, define a path from 0 to $2+2i$ which does not pass through $1+i$.

3. Define carefully what is meant by a *compact* subset of the plane. Show carefully that a compact subset of the plane is bounded.

4. For which real a is the function $u(x,y) = \cos^2 x \cosh^2 y + a \sin^2 x \sinh^2 y$ the real part of an entire function? For such a, find a possible imaginary part of the function.

5. Calculate the radius of convergence of

   (a) $\displaystyle\sum_{n=0}^{\infty} \frac{n+2}{3^n}(z-1)^n$ (b) $\displaystyle\sum_{n=1}^{\infty} \log(n) z^n.$

6. Give an explicit definition of a function $(\sin z)^z$ which is analytic in a neighbourhood of $z = \pi/2$. What can be said about the radius of convergence of the Taylor series of the function about $z = \pi/2$? What is the coefficient of $(z - \pi/2)$ in this Taylor series?

7. Use the generalised Cauchy Integral formula to evaluate the integral

$$\int_\gamma \frac{\sin z}{z^2 \cos 2z}\, dz$$

where γ is the circle $\{z : |z| = 1/2\}$.

8. Find the Laurent series about $z = 0$ of the function

$$\frac{1}{(z-1)(z+3i)}$$

for a domain which includes the point $z = -2$. .

9. (a) Give a formal definition for the function $z^{\frac{1}{2}}$ and describe where this function is defined. Show from your definition that $\left(z^{\frac{1}{2}}\right)^2 = z$.

The function f is given by

$$f(z) = \frac{z^{\frac{1}{2}} - 1}{z - 1}.$$

(b) f has a singularity at $z = 1$; what sort of singularity? Give a careful justification of your answer.

(c) What is the coefficient of the constant term in the Laurent series for f centered at $z = 1$?

10. A function f is analytic on the unit disc $\{z : |z| < 1\}$. For each natural number n, we have $f(1/n) = 2i/n^2$. Describe the function f. Give careful reasons for your answer. Hence give the derivative of f at $z = 1/2$.

11. Calculate, using the Residue Theorem,

$$\int_C \frac{z^2+1}{z(4z^2-1)}\, dz$$

where C is the circle with centre 0 and radius 1 described in the usual anti-clockwise direction.

12. Show the following using contour integration techniques. (You should indicate where you believe that certain integrals tend to zero but need not provide a proof.)

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+4)^2} = \frac{\pi}{16}.$$