The University of Melbourne

Semester 1 Assessment, 2007

Department of Mathematics and Statistics

620-221 Real and Complex Analysis

Instructions to Students:
All questions carry the same number of marks.

Identical Examination Papers: nil
Common content examinations: nil
Reading time: 15 minutes
Duration of examination: Three hours
Length of this question paper: 5 pages

Authorised materials:
Pens, rubbers, and rulers are authorised. No other materials are authorised; in particular, calculators are not authorised. Candidates are reminded that no written or printed material related to this subject may be brought into the examination. If you have any such material in your possession, you should immediately surrender it to an invigilator.

Instructions to Invigilators:
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1. (a) Find the argument of
$$\frac{(1+i)^7}{i^{19}(1-i)^{12}}.$$

Solution:
The argument is

$$\begin{aligned}
\arg\left(\frac{(1+i)^7}{i^{19}(1-i)^{12}}\right) &= \arg\left((1+i)^7\right) - \arg(i^{19}) - \arg\left((1-i)^{12}\right) \\
&= 7\arg(1+i) - 19\arg(i) - 12\arg(1-i) \\
&= 7\times\pi/4 - 19\times\pi/2 - 12\times -\pi/4 \\
&= -19\pi/4.
\end{aligned}$$

Thus the principal value of the argument is $-3\pi/4$.

(b) Give a geometrical interpretation of the inequality
$$|z+w| \leq |z| + |w|$$
where z and w are complex numbers.

Solution:
Consider the triangle formed by the points 0, z and w. The lengths of the sides are given by $|z|, |w|$ and $|z+w|$. Thus the inequality reflects the fact that *the sum of the lengths of two sides of a triangle exceeds the length of the third side.*

(c) Describe the set of points z on the complex plane satisfying $|z| < |z-2-2i|$.

Solution:
This describes the set of points z which are closer to the origin than to $2+2i$. It is therefore the half-plane bounded by the perpendicular bisector of $[0, 2+2i]$ and containing 0. Alternatively, the boundary can be described as the line passing through 1 and i.

2. (a) Explain carefully what is meant by an *open* subset of the plane. Give an example, with brief justification, of a subset of the plane which is neither open nor closed.

Solution:
The first sentence is bookwork. The closed unit disc, omitting the point 1, is neither open nor closed. It is not open as, for example, no neighbourhood of the point -1 lies entirely inside the disc. The complement of the disc is not open as no neighbourhood of 1 lies entirely inside the complement.

(b) Is the set consisting of a single point open or closed or neither? Give brief reasons.

Solution:
It is closed. If $\{z\}$ is the set and $w \neq z$ then a n'hood centered at w and of radius $|z-w|/2$ will not meet $\{z\}$ and so will lie entirely inside the complement. So the complement is open and $\{z\}$ is closed.

(c) Give a precise definition of a *path* in the complex plane. Using your definition, define a path from 0 to $2+2i$ which does not pass through $1+i$.

Solution:
Bookwork. Define $f : [0,4] \to \mathbb{C}$ by $f(t) = t$ if $0 \leq t \leq 2$ and $f(t) = 2 + (t-2)i$ if $2 \leq t \leq 4$.

3. Define carefully what is meant by a *compact* subset of the plane. Show carefully that a compact subset of the plane is bounded.

Solution:
Bookwork. Let S be a compact subset of the plane. Then the open subsets $D_r = \{z : |z| < r\}$ cover the plane if we allow r to take all positive real values. That is, $\mathbb{R}^2 = \cup_{r \in \mathbb{R}} D_r$. But then these sets also cover S and so, because S is compact, a finite number must cover S. Say

$$S \subseteq D_{r_1} \cup \cdots \cup D_{r_k}.$$

Set $t = \max\{r_1, \ldots, r_k\}$. Then $D_{r_j} \subseteq D_t$ and so $S \subseteq D_t$. That is, S is bounded.

4. For which real a is the function $u(x,y) = \cos^2 x \cosh^2 y + a \sin^2 x \sinh^2 y$ the real part of an entire function? For such a, find a possible imaginary part of the function.

Solution:
We have

$$\begin{aligned} u_x &= -2\cos x \sin x \cosh^2 y + 2a \sin x \cos x \sinh^2 y \\ &= -\sin 2x(\cosh^2 y - a \sinh^2 y). \end{aligned}$$

Thus

$$u_{xx} = -2\cos 2x(\cosh^2 y - a\sinh^2 y).$$

Also

$$\begin{aligned} u_y &= 2\cos^2 x \cosh y \sinh y + 2a \sin^2 x \sinh y \cosh y \\ &= \sinh 2y(\cos^2 x + a \sin^2 x) \end{aligned}$$

and so

$$u_{yy} = 2\cosh 2y(\cos^2 x + a\sin^2 x).$$

As, for the real part of an entire function, we must have $u_{xx} + u_{yy} = 0$ for all x and y, then we must have $\cos 2x = \cos^2 x + a\sin^2 x$ and $\cosh 2y = (\cosh^2 y - a\sinh^2 y)$. That is, we must have $a = -1$.

In this case, if v is the imaginary part, then $v_y = u_x = -\sin 2x \cosh 2y$ and $v_x = -u_y = -\sinh 2y \cos 2x$. Thus, up to a constant, $v(x, y) = -(1/2)\sin(2x)\sinh(2y)$.

5. Calculate the radius of convergence of

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{n+2}{3^n}(z-1)^n$ (b) $\displaystyle\sum_{n=1}^{\infty} \log(n) z^n.$

Solution:
(a) We use the ratio formula. The radius of convergence is

$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}}$$

when this limit exists. In this case, we have

$$\lim_{n\to\infty} \frac{\frac{(n+2)}{3^n}}{\frac{(n+3)}{3^{n+1}}} = \lim_{n\to\infty} 3 \times \frac{(n+2)}{(n+3)} = 3$$

(b) Note that $\log n < n$ and the series with terms $a_n z^n$ has radius of convergence 1. We can see this latter fact because the ratio test gives the radius of convergence as $\lim_{n\to\infty} n/n+1 = 1$. Thus the radius of convergence is at least 1. But, if $|z| \geq 1$ then the nth term $\log n|z|^n$ does not approach 0 with n and so the series cannot be convergent. Thus the radius of convergence is 1.

6. Give an explicit definition of a function $(\sin z)^z$ which is analytic in a neighbourhood of $z = \pi/2$. What can be said about the radius of convergence of the Taylor series of the function about $z = \pi/2$? What is the coefficient of $(z - \pi/2)$ in this Taylor series?

Solution:
We can define $f(z) = (\sin z)^z = \exp(z \operatorname{Log}(\sin z))$. This function is defined unless $\sin z$ is real and non-positive. But, if $z = x + iy$ then $\sin z = \sin z \cosh y + i \cos x \sinh y$ and this is real if $\cos x = 0$ or if $\sinh y = 0$. That is, if $\cos x = 0$ or if $y = 0$. It is therefore real and non-positive if either $\cos x = 0$ and $\sin x = -1$ or if $y = 0$ and $\sin x \leq 0$. That is, if either $x = 2n\pi - \pi/2$ or if $y = 0$ and $x \in [(2n-1)\pi, (2n)\pi]$ where n is integer but otherwise arbitrary.

Thus the definition gives a function which is analytic at $z = \pi/2$. The nearest singularity is either where $z = 0$ or where $z = \pi$. Thus the radius of convergence of the Taylor series is $\pi/2$. The coefficient of z in the Taylor series is given by $f'(\pi/2)$. We have

$$\begin{aligned} f'(z) &= (z \operatorname{Log}(\sin z))' \exp(z \operatorname{Log}(\sin z)) \\ &= \big(\operatorname{Log}(\sin z) + z\,(\operatorname{Log}(\sin z))'\big) \exp(z \operatorname{Log}(\sin z)) \\ &= \Big(\operatorname{Log}(\sin z) + z\frac{\cos z}{\sin z}\Big)) \exp(z \operatorname{Log}(\sin z)) \end{aligned}$$

and so $f'(\pi/2) = 0$.

7. Use the generalised Cauchy Integral formula to evaluate the integral

$$\int_\gamma \frac{\sin z}{z^2 \cos 2z}\, dz$$

where γ is the circle $\{z : |z| = 1/2\}$.

Solution:
Apply the (generalised) Cauchy Integral Formula with function $f(z) = \sin z / \cos 2z$. We have

$$f'(z) = \frac{\cos z}{\cos 2z} - \frac{(\sin z)(-2\sin 2z)}{\cos^2 2z} = \frac{(\cos z \cos 2z) - 2\sin z \sin 2z}{\cos^2 2z}.$$

Applying the CIF, we have

$$\int_\gamma \frac{f(z)\,dz}{z^2} = 2\pi i f'(0) = 2\pi i \left(\frac{1-0}{1}\right) = 2\pi i.$$

8. Find the Laurent series about $z = 0$ of the function

$$\frac{1}{(z-1)(z+3i)}$$

for a domain which includes the point $z = -2$. .

Solution:
Use partial fractions to write the function as

$$\frac{1}{1+3i}\left(\frac{1}{z-1} - \frac{1}{z+3i}\right).$$

Because we want the expansion to be valid for $z = -2$, we want to expand in powers of $1/z$ and $-z/(3i)$ respectively. So we rewrite the function again to be

$$\frac{1}{1+3i}\left(\frac{1}{z} \times \frac{1}{1-(1/z)} - \frac{1}{3i} \times \frac{1}{1-(-z/3i)}\right).$$

This then becomes

$$\frac{1}{1+3i}\left(\frac{1}{z} \times \sum_{j=0}^{\infty}(1/z)^j - \frac{1}{3i} \times \sum_{j=0}^{\infty}\left(\frac{-z}{3i}\right)^j\right) = \sum_{j=-\infty}^{j=+\infty} a_j z^j$$

where

$$a_j = \begin{cases} \frac{1}{1+3i} & \text{if } j < 0, \\ \frac{1}{1+3i}\left((-1)^{j+1} \times 1/(3i)^{j+1}\right) & \text{if } j \geq 0. \end{cases}$$

This expansion is valid for $|1/z| < 1$ and $|z/3i| < 1$; that is, $1 < |z| < 3$ and so includes $z = -2$.

9. (a) Give a formal definition for the function $z^{\frac{1}{2}}$ and describe where this function is defined. Show from your definition that $\left(z^{\frac{1}{2}}\right)^2 = z$.

The function f is given by

$$f(z) = \frac{z^{\frac{1}{2}} - 1}{z - 1}.$$

(b) f has a singularity at $z = 1$; what sort of singularity? Give a careful justification of your answer.

(c) What is the coefficient of the constant term in the Laurent series for f centered at $z = 1$?

Solution:
a) We define $z^{\frac{1}{2}} = \exp(\frac{1}{2}\operatorname{Log}(z))$. It will be defined on the cut plane avoiding the non-positive real axis. Note that

$$\left(z^{\frac{1}{2}}\right)^2 = \left(\exp(\frac{1}{2}\operatorname{Log}(z))\right)^2 = \exp(\operatorname{Log}(z)) = z.$$

b) We claim the singularity is removeable. We need to show that $\lim_{z\to 1} f(z)$ exists. But, we can write $z - 1 = (z^{\frac{1}{2}} - 1)(z^{\frac{1}{2}} + 1)$. Note that $z^{\frac{1}{2}} = 1$ only when $z = 1$. So, if $z \neq 1$, we have

$$\lim_{z\to 1} \frac{z^{\frac{1}{2}} - 1}{z - 1} = \lim_{z\to 1} \frac{1}{z^{\frac{1}{2}} + 1} = \frac{1}{2}.$$

Because this limit exists, the singularity is removeable.

c) The value of the constant term is the value of the above limit; that is, 1/2.

10. A function f is analytic on the unit disc $\{z : |z| < 1\}$. For each natural number n, we have $f(1/n) = 2i/n^2$. Describe the function f. Give careful reasons for your answer. Hence give the derivative of f at $z = 1/2$.

Solution:
The given information shows that f agrees with the function $g(z) = 2iz^2$ at the points $z = 1/n$. Since the set of points $\{1/n\}$ has a limit point at $z = 0$, the Identity Theorem implies that $f = g$. Hence, for all z in the unit disc, $f(z) = 2iz^2$. Thus $f'(z) = 4iz$ and so $f'(1/2) = 2i$.

11. Calculate, using the Residue Theorem,

$$\int_C \frac{z^2+1}{z(4z^2-1)}\,dz$$

where C is the circle with centre 0 and radius 1 described in the usual anti-clockwise direction.

Solution:
There are simple poles at $z = 0, z = \pm 1/2$. We can calculate the residue at each pole by calculating the limit of

$$\frac{2(z^2+1)}{(z(4z^2-1))'} = \frac{2(z^2+1)}{(12z^2-1))}.$$

The residues are $2/-1 = -2$ at $z = 0$ and $(5/2)/2 = 5/4$ at each of $z = \pm 1/2$. Thus, by the Residue Theorem, the integral is equal to

$$2\pi i \times (-2 + \frac{5}{4} + \frac{5}{4}) = \pi i.$$

12. Show the following using contour integration techniques. (You should indicate where you believe that certain integrals tend to zero but need not provide a proof.)

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+4)^2} = \frac{\pi}{16}.$$

Solution:
We use a contour Γ which is a semicircle, centered at the origin and with base $[-R, R]$ for large R and which is above the real axis. We will integrate the function $1/(z^2+1)^2$.

The singularities of the integrand are at points z where $z^2 = -4$. Thus $z = \pm 2i$ and only $z = 2i$ occurs within the semicircle. The singularity is a double pole. Thus we can calculate the residue as

$$\begin{aligned}
\lim_{z\to 2i} \frac{d}{dz}\left(\frac{(z-2i)^2}{(z^2+4)^2}\right) &= \lim_{z\to 2i} \frac{d}{dz}\left(\frac{1}{(z+2i)^2}\right) = \lim_{z\to 2i} \frac{-2}{(z+2i)^3} \\
&= \frac{-2}{(4i)^3} = \frac{-i}{32}.
\end{aligned}$$

Thus, by the residue theorem, we have

$$\int_{\Gamma} \frac{dz}{(z^2+4)^2} = 2\pi i \times \frac{-i}{32} = \frac{\pi}{16}$$

as long as Γ is big enough to contain the singularity($R > 1$).

But

$$\int_{\Gamma} \frac{dz}{(z^2+4)^2)} = \int_{-R}^{R} \frac{dx}{(x^2+4)^2)} + \int_{\Delta} \frac{dz}{(z^2+4)^2)}$$

where Δ is the curved part of the semicircle. As $R \to \infty$, the latter integral approaches 0 and the middle (real) integral approaches $\int_{-\infty}^{\infty} \frac{dx}{(x^2+4)^2}$. Thus we obtain

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+4)^2)} = \frac{\pi}{16}$$

as required.