00149

The University of Melbourne
Semester Examination, June 1999
Department of Mathematics
Subject Number: 620-221
Subject Title: Real and Complex Analysis

Exam Duration: Three (3) hours
Reading Time: Fifteen (15) minutes
This paper has three (3) pages

Authorized materials:
None.

Intructions to Invigilators:
Calculators are not allowed. Examination paper may be retained.

Instuctions to Students:
Full marks can be obtained by correctly answering all questions. Total marks 80.

Lodging of Paper with Baillieu Library:
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Beginning of exam paper

1 (a) Define the notion of a domain in the complex plane $\mathbf{C}$ and decide whether the following subsets of $\mathbf{C}$ are domains:

(1) $\{z : 3 < |z| < 4\}$, (2) the upper half plane $\{z : z = x + iy, x \in \mathbf{R}, y > 0\}$.

Indicate the reasons for your answers.

[6]

(b) State the Cauchy Convergence Criterion. Using it or otherwise, show that a series

$$\sum_{n=0}^{\infty} a_n$$

of complex numbers converges if

$$\sum_{n=0}^{\infty} |a_n|$$

converges.

[7]

2 (a) State and derive the Cauchy-Riemann equations.

[6]

(b) Show that if $f(z)$ is analytic in a domain and is real valued, then $f(z)$ is constant in that domain.

[7]

3 (a) Determine the radius of convergence of the following power series:

$$(1) \quad \sum_{n=0}^{\infty} nz^n, \qquad (2) \quad \sum_{n=0}^{\infty} (2-i)^n z^n.$$

[6]

(b) Show that

$$\mathrm{Log}\,(z) = (z-1) - \frac{(z-1)^2}{2} + \frac{(z-1)^3}{3} - \cdots$$

for $|z-1| < 1$, where $\mathrm{Log}\,(z)$ denotes the principal value of the logarithmic function. (Hint: Differentiate both sides.)

[7]

4 (a) State Cauchy's Integral Formula for the n^{th} derivative and evaluate

$$\int_\gamma \frac{e^{5z}}{(z-1)^n} dz,$$

where n is a positive integer and γ is the contour given by $\gamma(t) = 1+e^{it}, \quad 0 \leq t \leq 2\pi$. [8]

(b) State and prove Liouville's Theorem on bounded entire functions. [5]

5 (a) Evaluate

$$\int_\gamma \frac{1}{(z^2+1)} dz,$$

where γ is the contour given by $\gamma(t) = 1+3e^{it}, \quad 0 \leq t \leq 2\pi$. [7]

(b) State and prove the maximum modulus theorem for analytic functions. [7]

6 (a) Find the Laurent expansion of

$$\frac{(z+1)}{z(z-4)^3}$$

about the point $z = 4$. [7]

(b) Evaluate

$$\int_{-\infty}^{\infty} \frac{dx}{(x^4+1)}.$$

[7]

End of exam paper