SOME MODEL PROBLEMS FROM PARTS 4–7

(1) FIND RADIUS OF CONVERGENCE OF

$$\sum_{n=1}^{\infty} (1+i)^n z^n \qquad \sum_{n=0}^{\infty} \frac{z^{3n}}{2^n} \qquad \sum_{k=0}^{\infty} \frac{3^k}{4^k+5^k} z^k$$

(2) FIND ALL VALUES OF i^{2i} AND $2^{\pi i}$

(3) SOLVE $\quad \mathrm{Log}(z^2-1) = i\frac{\pi}{2}$

(4) SHOW $\quad \arccos(z) = -i \log\left(z + \sqrt{z^2-1}\right)$

WHERE BOTH log & $\sqrt{\ }$ are MULTIVALUED.

(5) STATE CAUCHY'S FORMULA FOR THE FIRST DERIVATIVE AND DEDUCE LIOUVILLE'S THEOREM.

(6) EVALUATE

$$\int_{C_R} \frac{e^{2z}}{(z-2)^n}\, dz$$

WHERE $C_R = 3e^{it}$ $\;0 \le t \le 2\pi$, $n \ge 1$ integer.

(7) DECIDE WHETHER $z\sin(z)$ HAS AN ANTIDERIVATIVE IN $\mathbb{C}$.

(1) $\sum_{n=1}^{\infty} (1+i)^n z^n$ $\qquad a_n = (1+i)^n \neq 0$ is coeff of z^n $\forall n$

$$\lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty} \left|\frac{(1+i)^n}{(1+i)^{n+1}}\right| = \lim_{n\to\infty} \frac{1}{|1+i|} = \frac{1}{\sqrt{2}}$$

$\Rightarrow R = \frac{1}{\sqrt{2}}$

or: $\frac{1}{R} = \limsup_{n\to\infty} \sqrt[n]{|a_n|} = \limsup_{n\to\infty} \sqrt{2} = \sqrt{2} \Rightarrow R = \frac{1}{\sqrt{2}}$

$\sum_{n=0}^{\infty} \frac{1}{2^n} z^{3n} = \sum_{n=0}^{\infty} a_k z^k \Rightarrow$ coeff of z^k is 0 if $3 \nmid k$

Coefficients: $1, 0, 0, \frac{1}{2}, 0, 0, \frac{1}{4}, 0, 0, \frac{1}{8}, 0, \ldots$
for $z^0, z^1, z^2, z^3, z^4, z^5, z^6, \ldots$

$$\Rightarrow \limsup_{k\to\infty} \sqrt[k]{|a_k|} = \limsup_{n\to\infty} \sqrt[3n]{a_{3n}}$$

$$= \limsup_{n\to\infty} \sqrt[3n]{\frac{1}{2^n}} = \limsup_{n\to\infty} \frac{1}{2^{1/3}} = \frac{1}{2^{1/3}}$$

$\Rightarrow R = 2^{1/3}$

$\sum_{k=0}^{\infty} \frac{3^k}{4^k+5^k} z^k$ $\qquad a_k = \frac{3^k}{4^k+5^k} \neq 0$ is coeff of z^k $\forall k$

$$\lim_{k\to\infty} \left|\frac{a_k}{a_{k+1}}\right| = \lim_{k\to\infty} \frac{3^k(4^{k+1}+5^{k+1})}{(4^k+5^k)3^{k+1}}$$

$$= \lim_{k\to\infty} \frac{1}{3} \frac{4\left(\frac{4}{5}\right)^k + 5}{\left(\frac{4}{5}\right)^k + 1} = \frac{1}{3} \frac{0+5}{0+1} = \frac{5}{3}$$

$\Rightarrow R = \frac{5}{3}$

(2) use "log" for the multivalued complex log. fct. and "$\log_{\mathbb{R}}$" for the single valued real log. fct.

$$i^{2i} = e^{2i\log(i)} = e^{2i\left(\log_{\mathbb{R}}|i| + i\left(\frac{\pi}{2}+2\pi k\right)\right)}$$

$$= e^{2i\left(0 + i\left(\frac{\pi}{2}+2\pi k\right)\right)}$$

$$= e^{-2\left(\frac{\pi}{2}+2\pi k\right)} = e^{-\pi(1+4k)} \qquad k \in \mathbb{Z}$$

[principal value if $k=0 \Rightarrow e^{-\pi}$]

$$2^{\pi i} = e^{\pi i \log(2)} = e^{\pi i\left(\log_{\mathbb{R}}(2) + i\,2\pi k\right)}$$

$$= e^{\pi i \log_{\mathbb{R}}(2) - 2\pi^2 k} \qquad k \in \mathbb{Z}.$$

[principal value if $k=0 \Rightarrow e^{\pi i \log_{\mathbb{R}}(2)}$

(3) $\quad \mathrm{Log}(z^2-1) = i\,\pi/2$

$$\Rightarrow z^2 - 1 = e^{\mathrm{Log}(z^2-1)} = e^{i\pi/2} = i$$

$$\Rightarrow z^2 = 1 + i = \sqrt{2}\, e^{i\pi/4} = 2^{1/2}\, e^{i\pi/4}$$

$$\Rightarrow z = \pm\, 2^{1/4}\, e^{i\pi/8}$$

(4) $\quad \arccos(z) = w$

$$\Rightarrow z = \cos(w) = \tfrac{1}{2}\left(e^{iw} + e^{-iw}\right)$$

$$\Rightarrow (e^{iw})^2 - 2z\,e^{iw} + 1 = 0$$

$$\Rightarrow e^{iw} = z + \sqrt{z^2-1}$$ where $\sqrt{\ }$ is 2-valued

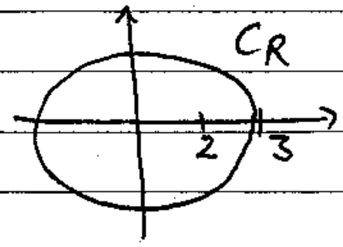
$$\Rightarrow iw = \log(z + \sqrt{z^2-1})$$ where log is multi-valued

$$\Rightarrow w = -i\,\log(z + \sqrt{z^2-1})$$

$$\Rightarrow \arccos(z) = -i\,\log(z + \sqrt{z^2-1})$$

(5) $\quad$ BOOKWORK

(6)



CAN APPLY CAUCHY'S FORMULA FOR THE DERIVATIVE WITH:

$f(z) = e^{2z}$ and $w = 2$ (CHECK!)

HAVE: $f^{(k)}(z) = 2^k e^{2z}$

SO:

$$\int_{C_R} \frac{e^{2z}}{(z-2)^n}\, dz = \int \frac{f(z)}{(z-w)^n}\, dz$$

$$= \frac{f^{(n-1)}(w)}{(n-1)!} \cdot 2\pi i = \frac{f^{(n-1)}(2)}{(n-1)!}\, 2\pi i$$

$$= \frac{2^{n-1} e^4}{(n-1)!}\, 2\pi i = \frac{2^n e^4}{(n-1)!}\, \pi i$$

(7) 3 VERSIONS

* $f(z) = z\sin(z)$ IS ANALYTIC IN $\mathbb{C}$, HENCE CAUCHY'S THM IMPLIES THAT $\int_C f(z)\,dz = 0$ FOR ALL CLOSED CONTOURS IN $\mathbb{C}$; BUT THIS IS EQUIVALENT TO $f(z)$ HAVING AN ANTIDERIVATIVE IN $\mathbb{C}$.

* NOTE THAT $\sin(z) - z\cos(z)$ IS AN ANTIDERIVATIVE FOR $z\sin(z)$

* FOR $\sin(z)$ HAVE A POWER SERIES WITH RADIUS OF CONVERGENCE $= \infty$.

  z is its OWN POWER SERIES WITH $R=\infty$

  $\Rightarrow$ CAN MULTIPLY THESE & THEN INTEGRATE TERM BY TERM, GIVING ~~THE~~ AN POWER SERIES FOR ~~THE~~ AN ANTI-DERIVATIVE W\ $R=\infty$.