SAMPLE QUESTIONS (FOR MST)

(1) EVALUATE $\sinh(\mathrm{Log}(i-1))$

(2) FIND RADIUS OF CONVERGENCE & DETERMINE BEHAVIOUR ON THE CIRCLE OF CONVERGENCE OF

(a) $\displaystyle\sum_{k=1}^{\infty} \frac{z^k}{k^2}$ (b) $\displaystyle\sum_{k=0}^{\infty} k z^{3k}$

(3) GIVE AN EXAMPLE OF A SUBSET OF $\mathbb{C}$ WHICH IS NEITHER OPEN NOR CLOSED <u>AND</u> INDICATE REASONS FOR YOUR ANSWER.

(4) STATE THE CAUCHY-RIEMANN EQUATIONS AND DECIDE WHETHER THE FUNCTION $f(z) = \overline{z}$ IS COMPLEX DIFFERENTIABLE AT $z=0$.

(5) (a) DEFINE CAUCHY-SEQUENCE & STATE THE CAUCHY CONVERGENCE CRITERION.

(b) SHOW THAT $z_n = 3 + \left(\frac{1+i}{6}\right)^n$ IS A CAUCHY-SEQUENCE.

(6) SHOW THAT $u(x,y) = x + 2xy$ IS HARMONIC IN $\mathbb{R}^2$.

SUPPOSE $f(z) = u(x,y) + i\,v(x,y)$ IS ANALYTIC IN $\mathbb{C}$. USE THE CAUCHY-RIEMANN EQUATIONS AND HARMONIC CONJUGATES TO FIND $f(z)$ <u>ONLY</u> EXPRESSED IN TERMS OF z.

(1) $\sinh(\mathrm{Log}(i-1)) = \frac{1}{2}\left(e^{\mathrm{Log}(i-1)} - e^{-\mathrm{Log}(i-1)}\right)$

$$= \frac{1}{2}\left(i-1-\frac{1}{i-1}\right)$$

$$= \frac{1}{2}\left(\frac{2(i-1)}{2} - \frac{i+1}{2}\right)$$

$$= \frac{i-3}{4}$$

(2) (a) $\sum_{k=1}^{\infty} \frac{z^k}{k^2}$

The coefficient of z^n is $\frac{1}{n^2}$ for all $n \in \mathbb{N}$.

$$\lim_{n\to\infty} \left| \frac{a_n}{a_{n-1}} \right| = \lim_{n\to\infty} \left| \frac{(n-1)^2}{n^2} \right| = \lim_{n\to\infty} \left| 1 - \frac{2}{n} + \frac{1}{n^2} \right| = 1$$

$\Rightarrow R = 1$ is the radius of convergence.

If $|z| = 1$ then: $\sum \left| \frac{z^k}{k^2} \right| = \sum \frac{1}{k^2}$ converges

$\Rightarrow \sum_{k=1}^{\infty} \frac{z^k}{k^2}$ converges absolutely on the circle of convergence.

(2) (b) $\sum_{k=0}^{\infty} k z^{3k}$

We can't use the ratio test because the coefficient a_n of z^n is 0 if $3 \nmid n$.

We therefore apply the Cauchy-Hadamard formula. (here: $a_n = |a_n|$).

if $3 \nmid n$: $\sqrt[n]{a_n} = 0$

if $3 | n$: $a_n = \frac{n}{3} \Rightarrow \sqrt[n]{a_n} = \frac{\sqrt[n]{n}}{\sqrt[n]{3}} \geq 1$ for $n \geq 3$

So $\frac{1}{R} = \limsup_{n \to \infty} \frac{\sqrt[n]{n}}{\sqrt[n]{3}} = \frac{1}{1} = 1 \Rightarrow R = 1$.

On the circle of convergence, $|z| = 1$

and: $|k z^{3k}| = k \longrightarrow \infty$ as $k \to \infty$

Hence the sequence does not converge for any point on the circle of convergence

(3) Consider the open interval $I = (0,1) \subset \mathbb{R} \subset \mathbb{C}$.

It is

- <u>not closed</u> (in $\mathbb{C}$): $\{\frac{1}{n}\}_{n\in\mathbb{N}}$ is a sequence in I,

  but $\lim_{n\to\infty} \frac{1}{n} = 0 \notin I$.

and

- <u>not open</u> (in $\mathbb{C}$): If I is open then there exists

  $\varepsilon > 0$ s.t. $B_\varepsilon(\frac{1}{2}) \subset I \subset \mathbb{R}$.

  But $\frac{1}{2} + \frac{\varepsilon}{2} i \in B_\varepsilon(\frac{1}{2})$

  and $\frac{1}{2} + \frac{\varepsilon}{2} i \notin \mathbb{R}$. ↯

(4) Let $z = x + iy$, and $f : \mathbb{C} \to \mathbb{C}$ be a complex function which we write as

$$f(z) = u(x,y) + i\, v(x,y)$$

with $u, v : \mathbb{R}^2 \to \mathbb{R}$.

Then the Cauchy-Riemann equations are

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.$$

For the second part:

$$f(z) = \bar{z} \quad \Rightarrow \quad f(z) = x - iy$$

so $u(x,y) = x$, $v(x,y) = -y$

and $u_x = 1$, $u_y = 0$, $v_x = 0$, $v_y = -1$.

Since $1 \neq -1$, we have $u_x \neq v_y$ and f is nowhere differentiable, and in particular not at $z = 0$.

5(a) A sequence $\{z_n\}$ is convergent

$\iff \forall \varepsilon > 0 \ \exists N(\varepsilon) \in \mathbb{N}$ s.t. $n, m \geq N(\varepsilon)$

implies $|z_m - z_n| < \varepsilon$

$\iff$: $\{z_n\}$ is a Cauchy sequence.

(b) $$|z_m - z_n| = \left| \left(\frac{1+i}{6}\right)^m - \left(\frac{1+i}{6}\right)^n \right|$$

$$\leq \left|\frac{1+i}{6}\right|^m + \left|\frac{1+i}{6}\right|^n \qquad (\Delta\text{-inequality})$$

$$= \left(\frac{\sqrt{2}}{6}\right)^m + \left(\frac{\sqrt{2}}{6}\right)^n$$

$r = \min(n, m)$

$$\leq 2\left(\frac{\sqrt{2}}{6}\right)^r$$

Now since $\frac{\sqrt{2}}{6} < 1$ we have $\lim_{r \to \infty} \left(\frac{\sqrt{2}}{6}\right)^r = 0$

so given $\varepsilon > 0$ $\exists N(\varepsilon)$ s.t.

$$\left(\frac{\sqrt{2}}{6}\right)^{N(\varepsilon)} < \frac{\varepsilon}{2}.$$

Also, if $r \geq N(\varepsilon)$ then:

$$2\left(\frac{\sqrt{2}}{6}\right)^r \leq 2\left(\frac{\sqrt{2}}{6}\right)^{N(\varepsilon)} < \varepsilon$$

(6) $u(x,y)$ is harmonic since:

$$\left.\begin{array}{ll} u_x = 1+2y, & u_{xx}=0 \\ u_y = 2x, & u_{yy}=0 \end{array}\right\} u_{xx}+u_{yy} = 0+0=0 \checkmark$$

To find the harmonic conjugates, use the Cauchy - Riemann equations and integrate:

$$v_y = u_x = 1+2y \to v = y+y^2+g(x)$$

$$v_x = -u_y = -2x \to v = -x^2 + h(y)$$

$$\to v = y+y^2-x^2+c, \quad c \in \mathbb{C} \text{ constant}$$

$$\Rightarrow f(z) = (x+2xy)+i(y+y^2-x^2+c)$$

$$= x+iy - i(x^2+2ixy-y^2) + ic$$

$$= x+iy - i(x+iy)^2 + ic$$

$$= z - iz^2 + ic$$