1. Introduction

The purpose of this document is to
(1) Define a representation of a $SL_2(\mathbb{R})$ on a Hilbert space $\mathcal{H}$.
(2) Give (some?) irreducible representations of $SL_2(\mathbb{R})$.
(3) State the Plancherel theorem on $SL_2(\mathbb{R})$.

The document is a product of some reading and chats between myself and Alex Amenta (Australian National University) during late November 2013. The main reference is ‘Representation Theory of Semisimple Groups’ by Knapp, and all references in square brackets [...] refer to this book.

2. Preliminaries

A complex Hilbert space $V$ is a complex vector space with an inner product $\langle -,- \rangle$ such that $V$ is complete with respect to the distance function
$$d(x,y) = \|x-y\| = \sqrt{(x-y,x-y)}.$$
A bounded linear operator on a complex Hilbert space $V$ is a linear transformation $L: V \longrightarrow V$ such that there exists $M > 0$ with
$$\|Lv\| \leq M\|v\|$$
for $v \in V$.

The operator norm $\|L\|$ of a bounded linear operator $L$ is the infimum of all such $M$. [§I.3] Let

$G$ be a topological group,
$V$ be a complex Hilbert space, and
$\mathcal{B}_{\text{inv}}(V)$ be the group of bounded linear operators on $V$ with bounded inverses.

A representation of $G$ on $V$ is a topological group homomorphism
$$\pi: G \longrightarrow \mathcal{B}_{\text{inv}}(V)$$
such that
(1) If $v \in V$ then the map
$$\phi_v: G \longrightarrow V$$
$$g \longmapsto \pi(g)v$$
is continuous at the point $g = I$.
(2) There exists $\varepsilon > 0$ and $K \in \mathbb{Z}_{\geq 1}$ such that
$$\|\pi(g)\| < K$$
for $g \in B(I,\varepsilon)$, where $B(I,\varepsilon)$ is the open ball with center $I$ and radius $\varepsilon$.  

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Let $C_c^\infty(G)$ be the ring of smooth compactly supported functions on $G$. Define

$$\pi': \ C_c^\infty(G) \rightarrow B_{\inv}(V)$$

where

$$\pi'(f)v = \int_G f(x)\pi(x)vdx$$

for $f \in C_c^\infty(G)$.

**Proposition 1.** The representation $\pi$ can be recovered from $\pi'$.

**Proof.** TO DO \hfill \blacksquare

Because of Proposition 1, we will sometimes abuse notation and write $\pi(f)$ instead of $\pi'(f)$, where $f \in C_c^\infty(G)$. So the domain of $\pi$ can be either $G$ or $C_c^\infty(G)$ depending on context.

An invariant subspace $U \in V$ of a representation $\pi$ is a vector subspace of $V$ such that

if $g \in G$ then $\pi(g)U \subseteq U$.

A representation $\pi$ is irreducible if it has no nontrivial closed invariant subspaces.

**Proposition 2.** If

$$L: V \rightarrow V$$

is a bounded linear operator then there exists a unique bounded linear operator

$$L^*: V \rightarrow V$$

such that

$$\langle Lx, y \rangle = \langle x, L^*y \rangle$$

for $x, y \in V$.

**Proof.** Riesz representation theorem. \hfill \blacksquare

The adjoint $L^*$ of a bounded linear operator $L$ is the operator $L^*$ from Proposition 2.

A unitary representation $\pi$ is a representation such that

$$\pi(g)\pi(g)^* = \pi(g)^*\pi(g) = I$$

for $g \in G$.

**Example 3** (§I.3). The map

$$\pi: SL_2(\mathbb{R}) \rightarrow B_{\inv}(L^2(\mathbb{R}^2))$$

defined by

$$\pi(g)f(x) = f(g^{-1}x)$$

for $f \in L^2(\mathbb{R}^2)$, is a unitary representation.

3. **Some irreducible representations of $SL_2(\mathbb{R})$**

The reference is [§II.5].
3.1. **Discrete series representations.** Let \( n \in \mathbb{Z}_{\geq 2} \). The \( n \)-dimensional Hardy space is the Hilbert space with vector space
\[
H_n^+ = \left\{ f : \mathbb{C} \to \mathbb{C} \mid f \text{ is analytic on the upper half plane, and } \|f\|^2 = \iint_{\Im z > 0} |f(z)|^2 y^{n-2} \, dx \, dy < \infty \right\}
\]
with inner product
\[
\langle f, g \rangle = \iint_{\Im z > 0} \frac{f(z)g(z)}{y^{n-2}} \, dx \, dy
\]
for \( f, g \in H_n^+ \).

The discrete series representations of \( SL_2(\mathbb{R}) \) are
\[
\mathcal{D}_n^+ : SL_2(\mathbb{R}) \to B_{\text{inv}}(H_n^+)
\]
where
\[
\mathcal{D}_n^+ \left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) f(z) = (-bz + d)^n f \left( \frac{az - c}{-bz + d} \right),
\]
and \( \mathcal{D}_n^- \) is defined 'by complex conjugation', from Knapp, for \( n \in \mathbb{Z}_{n \geq 2} \).

**Theorem 4.** The discrete series representations \( \mathcal{D}_n^+ \) and \( \mathcal{D}_n^- \) for \( n \in \mathbb{Z}_{n \geq 2} \) are irreducible unitary representations.

3.2. **Principle series representations.** The principle series representations of \( SL_2(\mathbb{R}) \) are
\[
\mathcal{P}^+_iv : SL_2(\mathbb{R}) \to B_{\text{inv}}(L^2(\mathbb{R}))
\]
\[
\mathcal{P}^-iv : SL_2(\mathbb{R}) \to B_{\text{inv}}(L^2(\mathbb{R}))
\]
defined by
\[
\mathcal{P}^+_iv \left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) f(x) = | -bx + d |^{1-iv} f \left( \frac{ax - c}{-bx + d} \right),
\]
\[
\mathcal{P}^-iv \left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) f(x) = \text{sgn}(-bx + d) \left[ \mathcal{P}^+_iv \left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) f(x) \right]
\]
for \( v \in \mathbb{R} \) and \( f \in L^2(\mathbb{R}) \).

**Theorem 5.** The principle series representations \( \mathcal{P}^+_iv \) and \( \mathcal{P}^-iv \) for \( v \in \mathbb{R} \) are irreducible unitary representations, with the exception of \( \mathcal{P}^-0 \).

4. **Trace class operators**

Let \( V \) be a complex Hilbert space with inner product \( \langle - , - \rangle \). A set of vectors \( \{ e_i \}_{i \in \mathbb{Z}_{\geq 1}} \) is a Hilbert basis [from Wolfram Mathworld, better reference?] of \( V \) if
(1) \( \langle e_i, e_j \rangle = \delta_{ij} \) for \( i, j \in \mathbb{Z}_{\geq 1} \) (orthonormality),
(2) If \( v \in V \) then there exists \( a_i \in \mathbb{C} \) for \( i \in \mathbb{Z}_{\geq 1} \) such that
  (a) \( \sum_{i \in \mathbb{Z}_{\geq 1}} |a_i|^2 < \infty \), and
  (b) \( \sum_{i \in \mathbb{Z}_{\geq 1}} a_ie_i = v \).
Let $L$ be a bounded linear operator on $V$ and let $\{e_i\}_{i \in \mathbb{Z}_{\geq 1}}$ be a Hilbert basis for $V$. The trace of $L$ is

$$\text{tr}(L) = \sum_{i \in \mathbb{Z}_{\geq 1}} \langle Le_i, e_i \rangle.$$ 

A trace class operator $L$ is a bounded linear operator such that there exists a Hilbert basis $\{e_i\}_{i \in \mathbb{Z}_{\geq 1}}$ for which $\text{tr}(L)$ converges.

**Proposition 6** (§X.1). If $L$ is a trace class operator then $\text{tr}(L)$ is independent of the choice of Hilbert basis.

5. **The Plancherel Theorem for SL$_2(\mathbb{R})$**

**Proposition 7.** The discrete series representations $D^+_n, D^-_n$ and principal series representations $P^+_iv, P^-iv$ of SL$_2(\mathbb{R})$ are trace class.

**Proof.** ???

**Theorem 8** (Plancherel Theorem, §II. 7, eqn (2.25)). If $h \in C_c^\infty(\text{SL}(2, \mathbb{R}))$ then

$$h(1) = \int_{-\infty}^{\infty} \text{tr}(P^+_iv(h))v\tanh\left(\frac{\pi v}{2}\right)dv + \int_{-\infty}^{\infty} \text{tr}(P^-iv(h))v\coth\left(\frac{\pi v}{2}\right)dv$$

$$+ \sum_{n=2}^{4} (n-1) \text{tr}(D^+_n(h) + D^-_n(h))$$

for a suitable normalisation of Haar measure.

6. **To Do**

1. Clean up the referencing.
2. Fill in the gaps.
3. Calculate the traces of the Discrete Series representations and the Principal Series representations.
4. What is the link to the representation theory of the Lie algebra $\mathfrak{sl}_2$?
5. Link with the ‘Plancherel Theorem’ from representation theory of finite groups, which says that the sums of the squares of the dimensions of the irreducibles of a group is equal to the cardinality of the group (apply this result to $S_n$ for an adorable combinatorial identity).