TWISTED CHEVALLEY GROUPS

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1. Introduction

The goals of this document are the following:

1. Construct Chevalley groups from Lie algebras, following [Ste67, §1 to §3].
2. Construct twisted Chevalley groups from Chevalley groups, following [Ste67, §11].

Goal (1) is covered in Sections 2 to 5. Goal (2) is covered in Section 6. The main reference is [Ste67]. The necessary prerequisites on Lie algebras can be found in [Ser87].

A running example will be:

1. Constructing the Chevalley group $GL_n(F_{q^2})$ from the Lie algebra $gl_n$.
2. Constructing the twisted Chevalley group $U_n(F_{q^2})$ from the Chevalley group $GL_n(F_{q^2})$.

2. Root Systems and Simple Systems

Let

$\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{C}$,
$\mathfrak{h}$ a Cartan subalgebra,
$\mathfrak{h}^* = \text{Hom}(\mathfrak{h}, \mathbb{C})$ the dual of $\mathfrak{h}$.

Fix $\alpha \in \mathfrak{h}^*$. The $\alpha$ weight space of $\mathfrak{g}$ is the set

$\mathfrak{g}^\alpha = \{ x \in \mathfrak{g} \mid [x, h] = \alpha(h)x, \text{ for } h \in \mathfrak{h} \}$.

A root is an $\alpha \in \mathfrak{h}^*$ such that $\mathfrak{g}^\alpha \neq 0$. The root system of $\mathfrak{g}$ is the set

$R = \{ \alpha \in \mathfrak{h}^* \mid \alpha \text{ is a root} \}$

The root space decomposition [Ser87, §VI.1.1] of $\mathfrak{g}$ is

$\mathfrak{g} = \mathfrak{h} \bigoplus \bigoplus_{\alpha \in R} \mathfrak{g}^\alpha$

A simple system for $R$ is a linearly independent set $S$ in the vector space $\mathfrak{h}^*$ such that if $\alpha \in S$ then

$\alpha \in \sum_{\alpha' \in S} \mathbb{R}_{\leq 0} \alpha' \quad \text{or} \quad \alpha \in \sum_{\alpha' \in S} \mathbb{R}_{\leq 0} \alpha'$

Date: August 27, 2013.
Proposition 1. Let \( \mathfrak{gl}_n \) be the Lie algebra of \( n \times n \) matrices over \( \mathbb{C} \) with the Lie bracket defined by \( [x, y] = xy - yx \) for all \( x, y \in \mathfrak{gl}_n \). Then \( \mathfrak{gl}_n \) has a Cartan subalgebra \( \mathfrak{h} \) consisting of all the diagonal matrices in \( \mathfrak{gl}_n \), so
\[
\mathfrak{h} = \mathbb{C}E_{11} \oplus \mathbb{C}E_{22} \oplus \ldots \oplus \mathbb{C}E_{nn},
\]
where \( E_{ij} \) is the matrix with 1 in the \((i, j)\)-entry and 0 elsewhere.

Proof. The set of diagonal matrices is a nilpotent and self-normalizing subalgebra of \( \mathfrak{gl}_n \). These two conditions are sufficient to show that \( \mathfrak{h} \) is a Cartan subalgebra, following the definition in [Ser87, §3.1]. □

Proposition 2. Let \( \mathfrak{h} \) be the Cartan subalgebra of \( \mathfrak{gl}_n \) given in Proposition 1. Let \( \epsilon_k \in \mathfrak{h}^* \) be the linear functional defined by \( \epsilon_k(E_{ll}) = \delta_{kl} \) for \( k, l \in \{1, 2, \ldots, n\} \). Then the root system of \( \mathfrak{g} \) corresponding to \( \mathfrak{h} \) is
\[
\Sigma = \{ \epsilon_i - \epsilon_j \mid i, j \in \{1, 2, \ldots, n\} \text{ and } i \neq j \}.
\]

Proof. We need to show:
(1) If \( \alpha \in \Sigma \) then \( \mathcal{L}_\alpha \neq 0 \).
(2) If \( \mathcal{L}_\alpha \neq 0 \) for some \( \alpha \in \mathcal{H}^* \) then \( \alpha \in \Sigma \).

Proof of (1): Let
\[
X = \begin{bmatrix}
x_{11} & x_{12} & \cdots & x_{1n} \\
x_{21} & x_{22} & \cdots & x_{2n} \\
& \vdots & \ddots & \vdots \\
x_{n1} & x_{n2} & \cdots & x_{nn}
\end{bmatrix} \in \mathcal{L}
\]
Note that
\[
E_{ij}E_{kl} = \delta_{jk}E_{il}
\]
and
\[
[E_{ij}, E_{kl}] = \delta_{jk}E_{il} - \delta_{li}E_{kj}
\]
for all \( i, j, k, l \in \{1, 2, \ldots, n\} \). Also, note that
\[
[X, H] = \alpha(H)X
\]
for all \( H \in \mathcal{H} \) if and only if
\[
[X, E_{kk}] = \alpha(E_{kk})X
\]
for all \( k \in \{1, 2, \ldots, n\} \). Also, we have
\[
[X, E_{kk}] = XE_{kk} - E_{kk}X
\]
\[
= \begin{bmatrix}
x_{1,k} & 0 & \cdots & -x_{k,k-1} & \cdots & -x_{k,n} \\
\vdots & \ddots & \ddots & \vdots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
x_{k+1,k} & \cdots & \cdots & \cdots & \cdots & x_{n,k}
\end{bmatrix}
\]
Let \( \epsilon_i - \epsilon_j \in \Sigma \). We claim that \( E_{ij} \in \mathcal{L}_{\epsilon_i - \epsilon_j} \).
For all $k \in \{1, 2, \ldots, n\}$ we have
\[
(\epsilon_i - \epsilon_j)(E_{kk})(E_{ij}) = (\delta_{ik} - \delta_{jk})(E_{ij}) = \begin{cases} E_{ij} & \text{if } i = k \\ -E_{ij} & \text{if } j = k \\ 0 & \text{otherwise} \end{cases}
\]
Hence
\[
[E_{ij}, E_{kk}] = (\epsilon_i - \epsilon_j)(E_{kk})(E_{ij})
\]
for all $k \in \{1, 2, \ldots, n\}$ and so $\mathcal{L}_{\epsilon_i - \epsilon_j} \neq 0$.

*Proof of (2):* Suppose
\[
\alpha = \alpha_1 \epsilon_1 + \alpha_2 \epsilon_2 + \ldots + \alpha_n \epsilon_n \in \mathcal{H}^*
\]
for some $\alpha_i \in \mathbb{C}$ with $\mathcal{L}_\alpha \neq 0$. Let
\[
X = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{bmatrix} \in \mathcal{L}_\alpha
\]
with $X \neq 0$. Then for all $k \in \{1, 2, \ldots, n\}$ we have
\[
\begin{bmatrix} x_{1,k} \\ \vdots \\ x_{k-1,k} \\ 0 \\ -x_{k,k+1} \\ \vdots \\ x_{n,k} \end{bmatrix} = \begin{bmatrix} \alpha_1 x_{11} & \alpha_1 x_{12} & \cdots & \alpha_1 x_{1n} \\ \alpha_1 x_{21} & \alpha_1 x_{22} & \cdots & \alpha_1 x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_1 x_{n1} & \alpha_1 x_{n2} & \cdots & \alpha_1 x_{nn} \end{bmatrix}.
\]
Suppose $k \in \{1, 2, \ldots, n\}$. If $\alpha_k \neq 0$ then deleting row $k$ and column $k$ of $X$ gives the zero matrix.

Furthermore, if $\alpha_k \neq 1$ and $\alpha_k \neq -1$ then $x_{l,k} = \alpha_k x_{l,k}$ and $x_{l,l} = -\alpha_k x_{l,l}$ for all $l \in \{1, 2, \ldots, n\}$, so $X = 0$, a contradiction. Hence for all $k \in \{1, 2, \ldots, n\}$ we have $\alpha_k \in \{-1, 0, 1\}$.

If $\alpha_k = 1$, then $x_{k,1} = x_{k,2} = \ldots = x_{k,n} = 0$.

If $\alpha_k = -1$, then $x_{k,1} = x_{k,2} = \ldots = x_{k,n} = 0$.

If $\alpha_k = 0$, then we have
\[
x_{k,1} = x_{k,2} = \ldots = x_{k,n} = 0 = x_{1,k} = x_{2,k} = \ldots = x_{n,k}
\]
extcept possibly $x_{kk} \neq 0$. Suppose $\alpha \neq \epsilon_i - \epsilon_j$ for all $i, j \in \{1, 2, \ldots, n\}$ with $i \neq j$.

Then there would be at least two positive coefficients of the $\epsilon_k$ or at least two negative coefficients of the $\epsilon_k$ in the expression

\[
\alpha = \alpha_1 \epsilon_1 + \alpha_2 \epsilon_2 + \ldots + \alpha_n \epsilon_n
\]
Suppose that the coefficients $\alpha_k$ and $\alpha_{k'}$ of $\epsilon_k$ and $\epsilon_{k'}$ are positive with $k \neq k'$, $k, k' \in \{1, 2, \ldots, n\}$. We show that this leads to a contradiction. A similar proof holds in the case of negative coefficients. Let $X \in \mathcal{L}_\alpha$ with $X \neq 0$. Then
\[
x_{k,1} = x_{k,2} = \ldots = x_{k,n} = 0 = x_{k',1} = x_{k',2} = \ldots = x_{k',n}
\]
Furthermore, deleting row \( k \) and column \( k \) of \( X \) gives the zero matrix, and also deleting row \( k' \) and column \( k' \) of \( X \) gives the zero matrix. The previous two facts combined force \( \mathcal{X} \) to be zero, contradicting \( \mathcal{X} \neq 0 \). Hence \( \alpha = \epsilon_i - \epsilon_j \) for some \( i, j \in \{1, 2, \ldots, n\} \) with \( i \neq j \).

A subset \( \Pi \subseteq \Sigma \) is called a simple system for \( \Sigma \) if [Ste67, pg. 266]:

1. \( \Pi \) is a linearly independent set.
2. Every root is a linear combination of the elements of \( \Pi \) in which all nonzero coefficients are either all positive or all negative.

**Example 3.** A simple system for the root system \( \Sigma \) of \( \mathcal{L} = \mathfrak{gl}_n \) is

\[
\Pi = \{\epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_3, \ldots, \epsilon_{n-1} - \epsilon_n\}
\]

**Proof.** Proof of (1). Suppose

\[
a_1(\epsilon_1 - \epsilon_2) + a_2(\epsilon_2 - \epsilon_3) + \ldots + a_{n-1}(\epsilon_{n-1} - \epsilon_n) = 0.
\]

for some \( a_1, a_2, \ldots, a_n \in \mathbb{R} \). Then

\[
a_1\epsilon_1 - a_1\epsilon_2 + a_2\epsilon_2 - a_2\epsilon_3 + \ldots + a_{n-1}\epsilon_{n-1} - a_{n-1}\epsilon_n = 0.
\]

Since \( \{\epsilon_1, \epsilon_2, \ldots, \epsilon_n\} \) is a linearly independent set, we have

\[
a_1 = a_2 = a_3 = \ldots = a_{n-1} = a_{n-2} = a_{n-1} = 0.
\]

Therefore

\[
a_1 = a_2 = \ldots = a_{n-1} = 0.
\]

So \( \Pi \) is a linearly independent set. Hence (1) is satisfied.

**Proof of (2).** Let \( \epsilon_i - \epsilon_j \in \Sigma \) with \( i, j \in \{1, 2, \ldots, n\} \) and \( i \neq j \).

If \( i < j \) then \( \epsilon_i - \epsilon_j = (\epsilon_i - \epsilon_{i+1}) + (\epsilon_{i+1} - \epsilon_{i+2}) + \ldots + (\epsilon_{j-1} - \epsilon_j) \).

If \( j < i \) then

\[
\epsilon_i - \epsilon_j = -(\epsilon_j - \epsilon_i)
\]

\[
= -(\epsilon_j - \epsilon_{j+1}) - (\epsilon_{j+1} - \epsilon_{j+2}) - \ldots - (\epsilon_{i-1} - \epsilon_i)
\]

Hence (2) is also satisfied.

**Weyl Groups.** The root system \( \Sigma \) of a Lie algebra \( \mathcal{L} \) generates \( \mathcal{H}^* \) as a vector space over \( \mathbb{C} \) [REFERENCE?]. Write \( V \) for \( \mathcal{H}^*_\mathbb{Q} \), the vector space over \( \mathbb{Q} \) generated by the roots. Then \( \dim_{\mathbb{Q}} V = \ell \), where \( \ell \) is the rank of the Lie algebra \( \mathcal{L} \).

Let \( \gamma \in V \). Since the Killing form is nondegenerate (?), there exists a \( H'_\gamma \in \mathcal{H} \) such that \( (H, H'_\gamma) = \gamma(H) \) for all \( H \in \mathcal{H} \), where \( (\cdot, \cdot) \) is (WHAT?). Define \( (\gamma, \delta) = (H'_\gamma, H'_\delta) \) for all \( \gamma, \delta \in V \) [Ste67, pp. 1-2].

For all nonzero \( \alpha, \beta \in V \), define

\[
\langle \alpha, \beta \rangle = \frac{2(\alpha, \beta)}{(\beta, \beta)}
\]

For each root \( \alpha \in \Sigma \), define a linear operator \( w_\alpha \in GL(V) \) by

\[
w_\alpha(p) = p - \langle p, \alpha \rangle \alpha
\]

for all \( p \in V \). The group generated by

\[
\{w_\alpha \mid \alpha \in \Sigma\}
\]

is called the Weyl group of \( \mathcal{L} \) [Ste67, pg. 2].
If \( \{\alpha_1, \alpha_2, \ldots, \alpha_\ell\} \) is a simple system of roots, then \( W \) is generated by the \( w_{\alpha_i} \) for \( i = 1, 2, \ldots, \ell \). Furthermore, every root is congruent under \( W \) to a simple root [Ste67, pg. 2].

**Example 4.** The Weyl group \( W \) of \( L = \mathfrak{gl}_n \) is generated by the matrices

\[
\begin{align*}
\begin{bmatrix}
0 & 1 \\
1 & 0 \\
& & \ddots \\
& & & 1
\end{bmatrix}, \\
\begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0 \\
& & & \ddots \\
& & & & 1
\end{bmatrix}, \\
\vdots \\
\begin{bmatrix}
1 & & & & \\
& \ddots & & & \\
& & 1 & & \\
& & & 0 & 1 \\
& & & 1 & 0
\end{bmatrix},
\end{align*}
\]

Hence \( W \cong \operatorname{Sym}(n) \).

3. Chevalley Bases

**Theorem 5.** [Ste67, pg. 6] There exists a choice of \( H_i \in \mathcal{H} \) with \( i \in \{1, 2, \ldots, \ell\} \) and \( X_\alpha \in \mathcal{L}_\alpha \) such that they form a basis for \( \mathcal{L} \) relative to which the defining relations are as follows:

1. \( [H_i, H_j] = 0 \)
2. \( [H_i, X_\alpha] = \langle \alpha, \alpha_i \rangle X_\alpha \)
3. \( [X_\alpha, X_{-\alpha}] = H_\alpha = \text{an integral linear combination of the } H_i \).
4. \( [X_\alpha, X_\beta] = \pm (r + 1)X_{\alpha + \beta} \text{ if } \alpha + \beta \text{ is a root, where } r \text{ is defined on } [\text{Ste67, pg. 3}] \).
5. \( [X_\alpha, X_\beta] = 0 \text{ if } \alpha + \beta \neq 0 \text{ and } \alpha + \beta \text{ is not a root.} \)

A basis chosen as in Theorem 5 is called a *Chevalley basis*. 
Example 6. The following matrices form a Chevalley basis for $L = \mathfrak{gl}_n$

\[
H_1 = \text{diag}(1, 0, \ldots, 0)
\]
\[
H_2 = \text{diag}(0, 1, \ldots, 0)
\]
\[
\vdots
\]
\[
H_\ell = \text{diag}(0, 0, \ldots, 1)
\]
\[
X_{\varepsilon_i - \varepsilon_j} = E_{ij} \text{ for all } i, j \in \{1, 2, \ldots, n\}, i \neq j
\]

4. CHEVALLEY GROUPS

Let $U$ be the universal enveloping algebra of $L$. [REFERENCE?]
Let $U_\mathbb{Z}$ be the $\mathbb{Z}$-subalgebra of $L$ generated by
\[
\left\{ \frac{X_\alpha^m}{m!} \mid m \in \mathbb{Z}_{\geq 1}, \alpha \in \Sigma \right\}
\]

For each root $\alpha \in \Sigma$, choose $X_\alpha \in L_\alpha$ as in Theorem 5. Let $V$ be a vector space over $\mathbb{C}$. A finitely generated (free abelian) subgroup $M$ of $V$ which has a $\mathbb{Z}$-basis which is a $\mathbb{C}$-basis for $V$ is called a lattice in $V$.

Example 7. $V = \mathbb{C}^n$, $M = \mathbb{Z}^n$.

[Ste67, pg. 13] From now on, let $V$ be a representation of $L$. [Ste67, pg. 20] Let $k$ be a field and define

\[V^k = M \otimes_\mathbb{Z} k\]

Example 8. For $V = \mathbb{C}^n$, $M = \mathbb{Z}^n$ we have $V^k = \mathbb{Z}^n \otimes_\mathbb{Z} k$.

We wish to study automorphisms of $V^k$ of the form $\exp(tX_\alpha) (t \in k, \alpha \in \Sigma)$ where

\[\exp(tX_\alpha) = \sum_{n=0}^{\infty} \frac{t^n X_\alpha^n}{n!}\]

[Ste67, §3] explains the above definition.

Write $x_\alpha(t)$ for $\exp(tX_\alpha)$ and $X_\alpha$ for the group $\{x_\alpha(t) \mid t \in k\}$. The group generated by all $X_\alpha (\alpha \in \Sigma)$ is called the Chevalley group constructed from $(L, V, M, k)$.

Example 9. Let $k = \mathbb{F}_q^2$ where $\mathbb{F}_q^2$ is a field with $q^2$ elements, $q$ a prime power. Let $V = \mathbb{C}^n$ be the standard representation of $L = \mathfrak{gl}_n$. Then

\[V^k = \mathbb{Z}^n \otimes_\mathbb{Z} \mathbb{F}_q^2 \cong \mathbb{F}_{q^2}^n\]

so that the action of $L$ on $V^k$ is the same as the action of $\mathfrak{gl}_n$ on $V$.

Fix $u \in \mathbb{F}_q^2$. Now

\[x_{\varepsilon_i - \varepsilon_j}(u) = \exp(uX_{\varepsilon_i - \varepsilon_j})\]
\[= I + uX_{\varepsilon_i - \varepsilon_j} + \frac{u^2 X_{\varepsilon_i - \varepsilon_j}^2}{2!} + \ldots\]
\[= I + uE_{ij} + \frac{u^2 E_{ij}^2}{2!} + \ldots\]
\[= I + uE_{ij}.\]
Therefore, the Chevalley group constructed from $(\mathfrak{gl}_n, \mathbb{C}^n, \mathbb{Z}^n, \mathbb{F}_{q^2})$ is generated by
\[ \{ I + tE_{ij} \mid t \in \mathbb{C}, i, j \in \{1, 2, \ldots, n\}; i \neq j \} \]
This group is called the general linear group $GL_n(\mathbb{F}_{q^2})$.

5. Automorphisms of Chevalley groups

**Definition 10.** Let $\Sigma \subseteq V$ and $\Sigma' \subseteq V'$ be root systems. A linear transformation $\tau : V \to V'$ is a morphism of root systems if
1. $\alpha \in \Sigma$ implies $\tau(\alpha) \in \Sigma'$.
2. $\langle \alpha, \beta \rangle = \langle \tau(\alpha), \tau(\beta) \rangle'$ for all $\alpha, \beta \in \Sigma$, where $\langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot \rangle'$ are the Cartan products on $V$ and $V'$ respectively.

Our definition of a morphism of root systems follows [Aut10] (Where is Steinberg’s definition?). Our Cartan product is
\[ \langle \alpha, \beta \rangle = \frac{2(\alpha, \beta)}{(\beta, \beta)} \]
for all $\alpha, \beta \in \Sigma$.

Let $G$ be a Chevalley group over a field $k$ of characteristic $p$.

**Proposition 11.** If $G$ is realised as a group of matrices and $\gamma$ is an automorphism of $k$, then the map $\gamma : x_\alpha(u) \mapsto x_\alpha(u^\gamma)$ on Chevalley generators extends to an automorphism $\theta$ of $G$.

**Definition 12.** An automorphism $\theta$ as in Proposition 11 is called a field automorphism [Ste67, pg. 168].

**Proposition 13.** Let $\tau$ be an angle preserving permutation of the simple roots $\Pi$ with $\tau \neq 1$. Assume all roots are equal in length. Then there exists an automorphism $\psi$ of $G$ and signs $\varepsilon_\alpha$ ($\varepsilon_\alpha = 1$ if $\alpha$ or $-\alpha$ is simple) such that
\[ \psi x_\alpha(u) = x_{\tau\alpha}(\varepsilon_\alpha u) \]
for all $u \in k$ and $\alpha \in \Sigma$.

**Definition 14.** An automorphism $\psi$ as in Proposition 13 is called a graph automorphism [Ste67, pg. 157].

**Definition 15.** Let $\sigma$ be an automorphism of a Chevalley group $G$ that is a composition of a field automorphism $\theta$ and a graph automorphism $\psi$. Also, suppose that, if $\rho$ is the corresponding permutation of the roots, then:
1. If $\rho$ preserves lengths, then $\text{ord}(\theta) = \text{ord}(\rho)$.
2. If $\rho$ does not preserve lengths, then $p\theta^2 = 1$, where $p$ is the map $x \mapsto x^p$.

Such an automorphism $\sigma$ is called a twisting automorphism [Ste67, my definition, see pg. 177].

6. Twisted Chevalley groups

**Definition 16.** Let $\sigma$ be a twisting automorphism of a Chevalley group $G$. The subgroup $G_\sigma$ of $G$ consisting of all elements in $G$ fixed by $\sigma$ is called a twisted Chevalley group.
Example 17. By Example ??, the Lie algebra \(\mathfrak{gl}_4\) has root system
\[
\Sigma = \{\epsilon_i - \epsilon_j \mid i \neq j; i, j \in \{1, 2, 3, 4\}\}.
\]
By Example 3, the root system \(\Sigma\) has simple system 3
\[
\Pi = \{\epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_3, \epsilon_3 - \epsilon_4\}
\]
Let \(F_{q^2}\) be a field with \(q^2\) elements where \(q\) is a prime power. Let \(G = GL_4(F_{q^2})\) be the Chevalley group constructed from \((\mathfrak{gl}_4, C^4, Z^4, F_{q^2})\). Let
\[
V' = \mathbb{Q}\Pi = \mathbb{Q}\{\epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_3, \epsilon_3 - \epsilon_4\}
\]
The linear transformation \(\tau : V \to V\) defined by
\[
\begin{align*}
\epsilon_1 - \epsilon_2 &\mapsto \epsilon_3 - \epsilon_4 \\
\epsilon_2 - \epsilon_3 &\mapsto \epsilon_2 - \epsilon_3 \\
\epsilon_3 - \epsilon_4 &\mapsto \epsilon_1 - \epsilon_2
\end{align*}
\]
is an automorphism of the root system \(\Sigma \subseteq V'\). By Proposition 13, the map \(\tau\) gives rise to a graph automorphism \(\psi\) of \(G\).

Define the Frobenius automorphism \(\gamma\) on \(F_{q^2}\) by \(u^{\gamma} = u^q\) for all \(u \in F_{q^2}\). By Proposition 11, the Frobenius automorphism gives rise to a field automorphism \(\theta\) of \(G\).

Let \(\sigma\) be the automorphism of \(G\) defined by
\[
\sigma = \psi \circ \theta
\]
The automorphism \(\sigma\) satisfies (1) and (2) in Definition 15, so \(\sigma\) is a twisting automorphism.

The twisted Chevalley group \(G_\sigma\) consisting of all elements of \(G\) fixed by \(\sigma\) is called the finite unitary group and is denoted by \(U_4(F_{q^2})\).

Next: Chevalley generators of \(U_4\) from Chevalley generators of \(GL_4\)?

References