What are the Kazhdan-Lusztig conjectures?

The main references are Williamson and Elias’ paper [WE12], Arun Ram’s talk on the aforementioned paper [Ram12], and Wikipedia [Wik13].

Let $W = \langle s_1, s_2, \ldots, s_n \mid (s_i s_j)^{m_{ij}} = 1 \rangle$ be a Coxeter group. The Hecke algebra $H$ of $W$ is the $\mathbb{Z}[v, v^{-1}]$-algebra generated by $T_{s_1}, T_{s_2}, \ldots, T_{s_n}$ with relations

$$T_{s_i}^2 = (v^{-1} - v)T_{s_i} + 1 \quad \text{and} \quad T_{s_i}T_{s_j} \cdots \underbrace{T_{s_i}T_{s_j} \cdots}_{m_{ij} \text{ factors}} = T_{s_j}T_{s_i}T_{s_j} \cdots \underbrace{T_{s_j}T_{s_i} \cdots}_{m_{ij} \text{ factors}}$$

**Theorem 1.** The Hecke algebra $H$ has a $\mathbb{Z}[v, v^{-1}]$ basis

$$\{T_x \mid x \in W\}$$

**Proof.** Theorem 4.4.6 of [GP00].

Let $- : H \to H$ be the $\mathbb{Z}$-algebra automorphism given by

$$\bar{T}_{s_i} = T_{s_i}^{-1} \quad \text{and} \quad \bar{v} = v^{-1}$$

The Kazhdan-Lusztig basis $\{C_x \mid x \in W\}$ of $H$ is characterised by

$$\bar{C}_x = C_x \quad \text{and} \quad C_x = T_x + \sum_{y < x} p_{y,x}(v)T_y$$

where $p_{y,x}(v) \in v\mathbb{Z}[v]$.

The polynomials $p_{y,x}(v)$ are called the Kazhdan-Lusztig polynomials.

**Conjecture 1.** The coefficients of the polynomials $p_{y,x}(v)$ are nonnegative.

**Conjecture 2.** For each $w \in W$, denote by $M_w$ the Verma module of highest weight $-w(\rho) - \rho$ where $\rho$ is the half-sum of the positive roots, and let $L_w$ be its irreducible quotient, the simple highest weight module of highest weight $-w(\rho) - \rho$. Then

1. $$\chi(L_w) = \sum_{y \leq w} (-1)^{\ell(w) - \ell(y)} p_{y,w}(1) \chi(M_y)$$
2. $$\chi(M_w) = \sum_{y \leq w} p_{w_0w, w_0y}(1) \chi(L_y)$$

where $w_0$ is the element of maximal length in $W$.

Conjecture 1 is found just before Definition 1.2 of [KL79], and was proven by Elias and Williamson in [WE12].

Conjecture 2 is Conjecture 1.5 of [KL79], and was proven independently by Beilinson and Bernstein [BB81] and Brylinski and Kashiwara [BK81]. Equations 1 and 2 are known to be equivalent.

Arun says: 'Historically, Conjecture 2 has been much more important than Conjecture 1 (so much so that the great majority of people working in the field didn’t even consider it worthwhile to put any real effort into Conjecture 1). There are indications that Conjecture 1 will lead to some further mathematical growth as the Elias-Williamson proof hints that there may be some geometry behind the "fake" Hodge theory that is their primary tool.'

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REFERENCES


