Generalised n-gons and the Feit-Thompson Theorem

Motivation

Classification theorem for finite simple groups (2004)

Let $G$ be a finite simple group. Then $G$ is isomorphic to one of the following:

- A cyclic group of prime power order.
- An alternating group of degree $\geq 5$.
- A simple group of Lie type.
- One of the 26 sporadic simple groups.

Groups of Lie type $\leftrightarrow$ Buildings

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Rank 2 buildings

Generalised n-gons.
Generalised n-gons

A generalised n-gon is a connected, bipartite graph with diameter \( n \) and girth \( 2n \) and such that each vertex is connected lies on at least two edges.

Examples: Generalised 3-gons:

\[
\begin{align*}
a) & \quad \text{Heawood graph} \\
b) & \quad \text{Type 1} \quad \text{Type 2}
\end{align*}
\]

Conversion into finite geometry: Call the type 1 vertices "points" and the type 2 vertices "lines". A point lies on a line if they share an edge on the graph.
a) becomes

\[ l_1 \cap l_2 = p_1 \]
\[ l_1 \cap l_3 = p_2 \]
\[ l_2 \cap l_3 = p_3 \]

b) becomes

(Fano plane)

Other examples:

- Tutte's 8-cage & generalised quadrangle
- Generalised 2-gons = complete bipartite graphs
- Generalised 3-gons = finite projective planes
- Generalised 4-gons = rank 2 polar spaces

Main theorem (Feit-Higman '64, Kilmoyer-Solomon '73):

If \( G \) is a thick, finite, generalised \( n \)-gon, then

\[ n \in \{2, 3, 4, 6, 8\} \]

\( G \) has \( > 2 \) vertices.
Outline of Kilmer-Casselman proof

Associate a Hecke algebra $A$ which "captures the symmetry" of $G$ (Meyr step).

The integer $n$ is somewhere inside the algebra $A$.

The representation theory of $A$ implies things like $\pi^{2n/n} \in \mathbb{C}$, where the result.

Copy details.

Let $V$ be the vector space spanned by the edges of $G$.

Let $c$ be an edge of $G$. Define linear transformations $T_1, T_2 : V \rightarrow V$ by

\[
T_1(c) = \sum_{d} \text{ is an edge going from } a \text{ type 1 vertex with } c \quad d
\]

\[
T_2(c) = \sum_{d} \text{ is an edge going from } a \text{ type 2 vertex with } c \quad d
\]
Lemma \( G \) thick \( \Rightarrow \) \( G \) regular

Type 1 vertices have the same degree,
Type 2 vertices have the same degree.

Proposition Let \( G \) be a finite thick generalized \( n \)-gon with \( q_1 + 1 \) polars on each point, \( q_2 + 1 \) points on each line, with \( q_1, q_2 \in \mathbb{Z}^+ \). Let \( A \) be the algebra generated by \( T_1 \) and \( T_2 \). Then \( \dim A = 2n \), and \( A \) has defining relations:

\[
T_1^2 = q_1 I + (q_1 - 1) T_1,
\]

\[
T_2^2 = q_2 I + (q_2 - 1) T_2,
\]

\[
\underbrace{T_1 T_2 T_1 \cdots}_{n \text{ terms}} = \underbrace{T_2 T_1 T_2 \cdots}_{n \text{ terms}}.
\]

Remark The algebra \( A \) is isomorphic to the Hecke algebra \( H \) of the Chevalley group in \( q_1 \) and \( q_2 \).
To finish off:

- Find all irreducible representations of $A$.
- Explicitly compute the characters $\chi$ of the irreducible representations.
- Define a bilinear form $(,) on the space of characters.
- Consider the fact that the character of $V$ as an $A$-module, $\chi_V$, is rational for all irreducible characters $\chi$, where the result.
Use algebraic tools to study finite geometry.

- Can already show

\[ q_1 \leq q_2 \leq q_1^2 \] \( q_1, q_2 \in \mathbb{Z}_{\geq 1} \)

- Nearest of generalised quadrilaterals with certain parameters:

\( (3, 6), (4, 11), (4, 12) \)

- Ovoids in the geometry \( U_4(2) \) are not known.

Main goal:

- Buildings

- Building diagrams

From Melbourne:

- Helebrato - Twisted Chevalley Groups,

- Quantum groups (Drinfeld),

- Hecke algebras

- Kazhdan-Lusztig theory