Schubert varieties and finite geometry

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Goal and motivation

Schubert calculus was first introduced to solve problems in enumerative projective geometry. Finite geometry is a modern descendent of enumerative projective geometry. The goal of this thesis is to make precise the relationship between modern Schubert calculus and finite geometry. The motivating problem is the existence of ovoids in the Hermitian variety \(\mathcal{H}(5, q^2)\), where \(q = p^i\) is a prime power.

Ovoids are \(n\)-dimensional generalisations of \(2\)-dimensional conics. In 2006, De Beule and Metsch\(^3\) showed that no ovoids exist in the Hermitian variety \(\mathcal{H}(5, 4)\). The general case \((q^2 > 4)\) is currently an open problem.

Finite geometry

An incidence structure \(G\) is a triple \((\mathcal{P}, \mathcal{B}, \mathcal{I})\) with \(\mathcal{I} \subseteq \mathcal{P} \times \mathcal{B}\).

The sets \(\mathcal{P}, \mathcal{B},\) and \(\mathcal{I}\) are called the points, blocks, and incidences respectively [4]. For example, the projective variety \(\mathbb{P}^n\) has an incidence structure \(G = (\mathcal{P}, \mathcal{B}, \mathcal{I})\) with \(\mathcal{P} = \{1\text{-dimensional vector subspaces of } \mathbb{F}^n\}\), \(\mathcal{B} = \{(n-1)\text{-dimensional vector subspaces of } \mathbb{F}^n\}\), \(\mathcal{I} = \{(p, b) \in \mathcal{P} \times \mathcal{B} | p \subseteq b\}\).

The Hermitian structure \(\mathcal{H}(n-1, q^2)\)

Let \(\mathbb{F}_q\) be the finite field with \(q^2\) elements, where \(q = p^i\) is a prime power. The Frobenius automorphism \(\pi : \mathbb{F}_q \rightarrow \mathbb{F}_q\) is the field automorphism defined by \(\pi = q^i\).

Let \(V = \mathbb{F}_q^n\) considered as an \(\mathbb{F}_q\)-vector space. Let \((\cdot, \cdot) : V \times V \rightarrow \mathbb{F}_q\) be the Hermitian form defined by \((v, w) = \sum v_i \overline{w_i}\). A totally isotropic subspace is a vector subspace \(W \subseteq V\) such that if \(v, w \in W\) then \((v, w) = 0\).

The Hermitian variety \(\mathcal{H}(n-1, q^2)\) has an incidence structure \(G = (\mathcal{P}, \mathcal{B}, \mathcal{I})\) where \(\mathcal{P} = \{V | p \subseteq V \text{ is a 1-dimensional totally isotropic subspace}\}\), \(\mathcal{B} = \{b \subseteq V | b \text{ is a maximal dimension totally isotropic proper subspace}\}\), \(\mathcal{I} = \{(p, b) \subseteq \mathcal{P} \times \mathcal{B} | p \subseteq b\}\).

Ovoids

An ovoid in an incidence structure \(G\) is a set of points \(\mathcal{O}\) such that if \(b \in \mathcal{B}\) is a block in \(G\) then \(b\) is incident with \(\mathcal{O}\) at exactly one point. For example, let \(G = (\mathcal{P}, \mathcal{B}, \mathcal{I})\) with \(\mathcal{P} = \{p_1, p_2, p_3, p_4\}\), \(\mathcal{B} = \{l_1, l_2, l_3, l_4\}\), \(\mathcal{I} = \{(p_i, l_i), (p_2, l_i), (p_3, l_4), (p_4, l_3), (p_4, l_4)\}\).

The open problem

Does there exist an ovoid in the Hermitian variety \(\mathcal{H}(5, q^2)\) for \(q^2 > 4\)?

Known results

Some of the following results can be found in [3] and [2]:

- Ovoids exist in \(\mathcal{H}(3, q^2)\) for all \(q\).
- There is a classification theorem for ovoids in \(\mathcal{H}(3, 4)\).
- No ovoids exist in \(\mathcal{H}(2n, q^2)\) for \(n \geq 2\).
- If no ovoids exist in \(\mathcal{H}(5, q^2)\) then no ovoids exist in \(\mathcal{H}(2n + 1, q^2)\) for \(n \geq 3\).
- No ovoids exist in \(\mathcal{H}(5, 4)\) (see [4]).

The \(B\)-coset definition of an ovoid

Let \(G\) be an incidence structure. Then there exists a Chevalley group \(G\) corresponding to \(G\). Each point \(p \in \mathcal{P}\) of \(G\) corresponds to a coset of a certain parabolic subgroup \(P_i\) of \(G\). Similarly, each block \(b \in \mathcal{B}\) of \(G\) corresponds to a coset of a certain parabolic subgroup \(P_j\) of \(G\). An ovoid in \(G\) is a set of points \(\{gP_1, gP_2, \ldots, gP_j\}\) of \(G\) such that

\[
\bigcup_{i \in I(2^n, n)} gP_i = G
\]

and the union is disjoint.

Motivation for introducing Schubert varieties

The following problems motivate the study of Schubert varieties [1]:

- How many lines meet four fixed lines?
- How many lines meet four given curves \(C_1, C_2, C_3, C_4 \subseteq \mathbb{P}^n\)?
- How many conics in \(\mathbb{P}^2\) are tangent to five given conics?

These are problems about incidence structures!

Schubert varieties

Let \(G\) be a Chevalley group over \(\mathbb{F}_q\), where \(\mathbb{F}_q\) is a finite field with \(q\) elements. Then \(G\) has a Bruhat decomposition

\[
G = \bigcup_{w \in W} BwB,
\]

where \(B\) is a Borel subgroup and \(W\) is the Weyl group of \(G\). Let \(w \in W\). The Schubert variety corresponding to \(w\) is

\[
X_w = \bigcup_{v \leq w} BvB,
\]

where \(\leq\) is the Bruhat order on \(W\).

Current work in progress

For the Hermitian variety \(\mathcal{H}(n-1, q^2)\), the corresponding (twisted) Chevalley group is \(U(n, \mathbb{F}_q)\). The references [7] and [6] are being used to make an appropriate presentation of \(U(n, \mathbb{F}_q)\) so that the Schubert variety theory works. Afterwards, the theory of Schubert varieties will be used to answer the problem of existence of ovoids in the Hermitian variety \(\mathcal{H}(5, q^2)\).

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References