Senior Solutions

1. Find all integer values of \( n, 90 \leq n \leq 100, \) that can not be written in the form \( a + b + ab \) where \( a \) and \( b \) are positive integers. [Note: 0 is not a positive integer.]

Solution.

The statement \( n = a + b + ab \) is equivalent to \( n = (1 + a + b + ab) - 1 \) or \( n = (1 + a)(1 + b) - 1, \) or \( n + 1 = (1 + a)(1 + b), \) so that \( n + 1 \) is a product of these two numbers. The statement that \( a \) and \( b \) are positive is equivalent to saying that the two factors \( 1 + a \) and \( 1 + b \) are both at least 2. So \( n \) can be written as \( a + b + ab \) precisely when \( n + 1 \) is a composite number, and can not be when \( n + 1 \) is prime. The prime numbers from 91 to 101 are 97 and 101, so the answer is \( n = 96 \) and 100.

2. There is a warehouse containing \( k \) rooms in a row, where \( k \) is a positive integer. In the first room there are exactly \( 1 \times k \) boxes, in the second room there are \( 2 \times (k - 1) \) boxes, and so on, up to the last with \( k \times 1. \)

(a) Show that the total number of boxes in the warehouse is even when \( k = 1000 \) and odd when \( k = 1001. \)

(b) Show that for all \( k \) the total number of boxes is \( \frac{k(k+1)(k+2)}{6}. \) You may use the formulae \( 1 + 2 + \cdots + k = \frac{k(k+1)}{2} \) and \( 1^2 + 2^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6}. \)

Solution.

(a) The number of boxes in room \( i \) is \( i \times (k + 1 - i). \) If \( k \) is even, then this product is even because \( k + 1 - i \) is even whenever \( i \) is odd. Hence the total number of boxes in all the rooms is even when \( k \) is even, for example when \( k = 1000. \) On the other hand, if \( k \) is odd, then \( k + 1 - i \) is odd whenever \( i \) is odd. So the number of boxes in room \( i \) is odd whenever \( i \) is odd. For \( k = 1001 \) there are exactly 501 odd numbers from 1 to \( k \) inclusive, and therefore an odd number of rooms with an odd number of boxes. So the total number of boxes is odd.

(b) The total number of boxes is

\[
1 \times k + 2 \times (k - 1) + \cdots + k \times 1 = 1 \times (k - 0) + 2 \times (k - 1) + \cdots + k \times (k - 1) - (1 \times 0 + 2 \times 1 + \cdots + k \times (k - 1)).
\]

Now note that

\[
(1 \times k + 2 \times k + \cdots + k \times k) = k \times (1 + 2 + \cdots + k) = \frac{k(k+1)}{2} \times k
\]

and

\[
1 \times 0 + 2 \times 1 + \cdots + k \times (k - 1) = 1 \times (1 - 1) + 2 \times (2 - 1) + \cdots + k \times (k - 1) = 1 \times 1 + 2 \times 2 + \cdots + k \times k - (1 \times 1 + 2 \times 1 + \cdots + k \times 1) = 1^2 + 2^2 + \cdots + k^2 - (1 + 2 + \cdots + k) = \frac{k(k+1)(2k+1)}{6} - \frac{k(k+1)}{2}.
\]
So the total number is
\[
\frac{k(k+1)}{2} \times k - \frac{k(k+1)(2k+1)}{6} + \frac{k(k+1)}{2} = \frac{3k(k+1)(3k-(2k+1)+3)}{6} = \frac{k(k+1)(k+2)}{6}.
\]

3. Show that if \(a\) and \(b\) are real numbers and \(a^2 + ab + b^2 = 0\), then \(a = b = 0\).

**Solution.**
Multiplying the given equation by \((a-b)\) gives \(a^3 - b^3 = 0\), and so \(a^3 = b^3\) and then \(a = b\). Substituting \(b = a\) into the original equation gives \(3a^2 = 0\), and so \(a^2 = 0\) so \(a = 0\). From \(b = a\) we get \(b = 0\) also.

4. Given \(0 \leq x \leq 1\) and \(0 \leq y \leq 1\), show that \(\frac{x}{1+y} + \frac{y}{1+x} \leq 1\).

**Solution.**
Without loss of generality we can assume that \(x \leq y\) (otherwise, the same argument will apply with \(x\) and \(y\) interchanged). Hence, multiplying by the non-negative quantity \(x\), we get \(x^2 \leq xy\). Also since \(0 < y < 1\), \(y^2 \leq 1\). Adding these two inequalities gives
\[x^2 + y^2 \leq xy + 1.\]
Add \(x + y\) to both sides and factorise: \(x(1+x) + y(1+y) \leq (1+x)(1+y)\). Dividing both sides of this inequality by the (positive) quantity \((1+x)(1+y)\) yields the desired inequality.

5. A quadrilateral \(PQRS\) has sides \(SP = 2\), \(PQ = 1\), \(RS = \sqrt{2} QR\), the angle at \(P\) is 90° and the diagonal \(PR = 2\). The point \(T\) lies on \(SP\) with angle \(RTP = 90°\). Find the length \(TP\).

**Solution.**
Let \(x\) denote \(TP\). Then \(ST = 1-x\) and so Pythagoras’ Theorem applied to triangle \(STR\) gives \(RT = \sqrt{4-x^2}\). Letting \(H\) denote the foot of the perpendicular from \(R\) to the line \(PQ\), since \(TRHP\) is a rectangle we have \(RQ = x\) and \(HP = RT = \sqrt{4-x^2}\). Hence \(HQ = \sqrt{4-x^2} - 1\). Again applying Pythagoras to triangle \(QRH\),
\[
(QR)^2 = x^2 + (\sqrt{4-x^2} - 1)^2
= x^2 + 4 - x^2 - 2\sqrt{4-x^2} + 1
= 5 - 2\sqrt{4-x^2}.
\]
Hence \((SR)^2 = (Q\sqrt{2}QR)^2 = 10 - 4\sqrt{4-x^2}\).
Finally applying Pythagoras’ Theorem to triangle \(SRQ\) gives
\[
(SR)^2 = (RT)^2 + (TS)^2, \quad \text{or} \quad 10 - 4\sqrt{4-x^2} = (\sqrt{4-x^2})^2 + (2-x)^2
\]
which simplifies to \(1 + 2x = 2\sqrt{4 - x^2}\). Squaring both sides of this equation, we get

\[
1 + 4x + 4x^2 = 4(4 - x^2), \quad \text{or} \quad 8x^2 + 4x - 15 = 0.
\]

Solving this last equation using the quadratic formula gives \(x = -\frac{1}{4} \pm \frac{1}{4} \sqrt{31}\). The negative root gives a negative answer which is not possible, so we deduce \(TP = \frac{1}{4}(\sqrt{31} - 1)\).

6. Let \(C\) be a circle, and \(C_1\) and \(C_2\) two nonintersecting circles inside and touching \(C\) at points \(A\) and \(B\) respectively. Let \(t\) be a common tangent of \(C_1\) and \(C_2\) touching them at points \(D\) and \(E\) respectively, such that both \(C_1\) and \(C_2\) are on the same side of \(t\). Let \(F\) be the point of intersection of the lines through \(AD\) and \(BE\). Show that \(F\) always lies on \(C\).

Solution.

For a minute forget, about \(C_2\). Let \(O\) and \(o\) be the centres of \(C\) and \(C_1\) respectively, and let \(D'\) and \(E'\) be the points of intersection of the line \(t\) with the circle \(C\), and let \(F'\) be the point of intersection of the perpendicular bisector of the interval \(C'D'\) with the line \(AD\). We will show that \(F'\) always lies on \(C\). Since this is for \(any\) circle \(C_1\) inside \(C\) and just touching \(t\) and \(C\), it also applies to \(C_2\); that is, the point \(F'\) for \(C_2\) is also on \(C\). This implies that \(F = F'\), and so \(F\) must lie on \(C\).

It only remains to show that \(F'\) lies on \(C\). First note that \(O\) lies on the perpendicular bisector of the interval \(E'F'\) because \(E'F'\) is a chord of the circle \(C\) (the bisector also bisects \(C\)). It follows that triangle \(AoD\) is similar to triangle \(AOF'\) because

(i) they have the common angle \(OAF'\);

(ii) \(oD\) is parallel to \(OF'\) because both are perpendicular to \(D'E'\) (in the case of \(oD\), this is because \(C_1\) is tangent to \(t\) at \(D\)). Thus angle \(AoD'\) equals angle \(AOF'\).

But triangle \(AoD'\) is isosceles, as \(Ao\) and \(D'o\) are both radii of the circle \(C_1\). So \(AOF'\) must also be isosceles, and so \(AO = OF'\). Hence \(F'\) lies on the circle \(C\), as required.
7. 32 stones are placed, one in each square of the first four rows of an $8 \times 8$ chessboard arranged as 8 horizontal rows, one above the other, as in the figure. A game is played in which each move consists of choosing two adjacent squares $X$ and $Y$ in the same row each containing at least one stone, with an empty square $Z$ in the next row directly above either $X$ or $Y$, then transferring all the stones in $X$ and in $Y$ to $Z$. Prove that no square of the last (eighth) row can ever be occupied by any stones.

Solution.
Let's treat the squares as real estate. Each occupied square in the first row has value $\$1$ each, in the second row each occupied square has value $\$2$, and so on, doubling the value every time we move up a row. All unoccupied squares have no value. The occupied squares in the fourth row then have value $\$8$ each, so the total value of real estate at the start of the game is $\$8 \times (1 + 2 + 4 + 8) = \$120$. Note that every time a move occurs, and the stones from two squares are transferred up one row, two occupied squares on one row become unoccupied, whilst one unoccupied square on the next row gets occupied, so the total value of real estate must remain the same. So it remains at $\$120$ throughout the game. On the other hand, if a square in the last row were occupied, it would have value $2^7 = 128$. This is too much, so it can never happen.