1. A circular table is standing in one corner of a rectangular room, touching two walls. A fly is sitting on top of the table 16 cm from one wall and 18 cm from the other wall. What is the maximum possible diameter of the table? What is the minimum possible diameter of the table?

Solution: A moment’s consideration of the geometry of the situation is sufficient to see that, in both cases, the fly is sitting on the perimeter of the table. In one case (the largest possible table) as close to the wall as possible, and in the other (the smallest table) as far away as possible. Assume the corner of the room is in the second quadrant of the Cartesian coordinate system with origin at the centre of the table. In the former case, denoting the radius of the table by \( r \), the fly’s coordinates are \((18 - r, r - 16)\). Then by Pythagoras’s theorem we have

\[
r^2 = (18 - r)^2 + (r - 16)^2,
\]

which expands to \( r^2 - 68r + 580 = 0 \), which has solutions \( r = 58 \) or \( r = 10 \). The latter solution is clearly unphysical, so the required diameter is 116 cm. For the minimum diameter, the fly’s coordinates are now \((18 - r, 16 - r)\), and by Pythagoras’s theorem we obtain the same equation as above. In this case we need the second solution, \( r = 10 \), so the minimum diameter is 20 cm.

2. Three runners, Tony, Nick and Mary participated in a race. Mary got held up at the start and began running last, while Nick was second from the start. During the race Mary exchanged positions with other contestants 6 times, while Tony did that 5 times. It is known that Nick finished ahead of Tony. In what order did they finish?

Solution: If you start in a given position and swap positions an even number of times, you will finish in the same position modulo 2. (That is, if you start second, swap an even number of times, you must finish second, or fourth, or sixth,...). Similarly, if you switch places an odd number of times, you will finish in an odd position (no pun intended.)

Now, Tony started first, and exchanged positions 5 times. He therefore finished in an even position, which must be second. As Nick finished ahead of Tony, he won, and Mary came last. (As a check, note that Mary started in 3rd position, and finished in 3rd position, so must have swapped places an even number of times, which she did.) (A number of people tried to provide a “proof by example.” This is only a valid method of proof if all possibilities are taken into account - which is why it is so seldom used!)

3. Find (without using calculus) the straight line through the point \((3, 4)\) which cuts out minimal area in the first quadrant.

Solution: The key to this question is the inequality that says the arithmetic mean of a set of positive numbers is greater than or equal to their geometric mean, with equality if and only if the numbers are all equal.

Let the line cut the \( y \) axis at \( y = y_1 = 4 + a \) and the \( x \) axis at \( x = x_2 = 3 + b \). The area to be minimised is the sum of the upper triangle (area \( 3a/2 \)) plus the included rectangle (of fixed area 12 sq. units) plus the bottom triangle (area \( 4b/2 \)). But \( a \) and \( b \) are not independent, as the two triangles are similar. Hence \( a/3 = 4/b \). Hence we must minimise \( 3a/2 + 12 + 4b/2 = 18/b + 12 + 2b \).

Now \( 18/b + 2b \) is just the arithmetic mean of \( 36/b \) and \( 4b \), which is greater than or equal to their geometric mean \( \sqrt{36/b \times 4b} = \sqrt{144} = 12 \). The minimum is therefore 12, which occurs when the two terms are equal, that is when \( 36/b = 4b \) or \( b = 3 \). Hence the line in question passes through \((3, 4)\) and \((6, 0)\), and hence has equation \( y = -\frac{4}{3}x + 8 \).
If you don’t know the inequality, the problem can still be solved (and maybe more easily) by completing the square. To minimise \(18/b + 12 + 2b\) write it as
\[
2(\sqrt{b} - 3/\sqrt{b})^2 + 24,
\]
which clearly is \(\geq 24\), with equality occurring when
\[
(\sqrt{b} - 3/\sqrt{b}) = 0,
\]
or \(b = 3\). The rest of the solution is as above.

4. A retired farmer is selling his land to his neighbours, but retaining the farmhouse as his retirement home. The whole property forms a square, including the smaller square plot in the corner which contains the farmhouse and is not for sale. The three fields labelled A, B and C are rectangular, all the same shape but different sizes. The farmer is asking the same price per hectare for each. If he wants $10,000 for field B, how much does he expect for the three together?

Solution: Denote the long side of field B by \(b\), the short side by \(e\). Denote the short side of field C by \(f\), then the long side is of length \(e - f\) (as both sides of the house block are of this length). Then the long side of field A is of length \(b + ca - f\) and the short side is of length \(b - f\). Since the fields are of similar shape, we have, from fields B and C, that
\[
\frac{e}{b} = \frac{f}{e - f},
\]
and from B and A, that
\[
\frac{e}{b} = \frac{b - f}{b + e - f}.
\]
Eliminating \(f\) from these two equations gives \(3e^2 = b^2\) or \(\frac{b}{e} = \sqrt{3}\). From the first of the above equations it then follows that \(\frac{b}{e} = 1 + \sqrt{3}\). Multiplying these two gives \(\frac{b}{f} = 3 + \sqrt{3}\).

Now, we can answer the question. We require the ratio of the sum of the areas of fields A, B and C to that of field B, which must be multiplied by $10,000 to get the required price. That ratio is
\[
\frac{eb + (b-f)(b+e-f) + f(e-f)}{eb} = 1 + \left(1 - \frac{f}{b}\right)(1 + \frac{b}{e} - \frac{f}{e}) + \frac{f(e-f)}{eb}.
\]
Substituting the known values for the ratios in the above equation gives, after a little simplification, the result 3. Hence the price obtained was $30,000.

A solution requiring somewhat less algebra was provided by Edward Shin. Denote the long and short sides of field B by \(k\) and \(y\) respectively. Denoting the long side of field A by \(x\) the short side is of length \(x - y\), and the house is on a block with side length \(x - k\). The lengths of the long and short sides of field C are then \(x - k\) and \(y + k - x\) respectively. The area of the three fields A,B, and C is then \(x^2 - (x - k)^2\). As costs are proportional to area, we may assume that each square
unit costs $1. Then the cost of B is $ky = 10,000. Since the shapes of A, B and C are the same, we have
\[
\frac{y}{k} = \frac{x - y}{x} = \frac{y + k - x}{x - k}.
\]
The first equality gives \(xy = kx - ky = kx - 10,000\), while equating the first and third term above gives \(y(x - k) = k(y + k - x)\), or \(xy = k^2 - kx + 20,000\). Equating the two expressions for \(xy\) gives \(2kx - k^2 = 30,000\). This is the area sought, hence the cost is $30,000.

5. We define a domino to be a 2 \times 1 tile with no markings on the surface — so all dominos are equivalent. To cover a 2 \times n rectangle with dominos, it is easy to see that, when \(n = 1\) there is only one tiling:

\[
\begin{array}{c}
\end{array}
\]

Also, when \(n = 2\) there are 2 tilings:

\[
\begin{array}{c|c}
\end{array}
\]

and when \(n = 3\) there are 3 tilings:

\[
\begin{array}{c|c|c}
\end{array}
\]

Now imagine we have an eccentric domino tiler who specialises in tilings of the 2 \times n rectangle, and pays $4 for each vertical domino in a tiling and $1 for each horizontal domino. How many tilings are worth exactly $m by this criterion? For example, when \(m = 6\) there are exactly three solutions, shown below:

\[
\begin{array}{c|c|c}
\end{array}
\]

Solution: For problems of this sort, enumerating the first few cases is often illuminating. In this way, we quickly see that there is no solution for \(m = 1\), there is exactly 1 solution for \(m = 2\) (two horizontal dominos stacked on top of each other), no solution for \(m = 3\), two solutions for \(m = 4\) (1 vertical domino, and a doubling of the pattern for \(m = 2\).) By this time it will come as no surprise to find no solutions for \(m = 5\) and you should be able to make an argument for there being no solutions for \(m\) odd (it requires an odd number of horizontal tiles, which cannot then tile the rectangle.) We are given the solution for \(m = 6\), and you should be able to draw all 5 solutions with \(m = 8\). Now look at the tilings. It is clear that tilings for \(m = 2n + 4\) are produced by adding a pair of stacked horizontal dominos to the left of all tilings with \(m = 2n + 2\), as well as by adding a single vertical domino to the right of all tilings with \(m = 2n\).

Denoting the number of solutions with \(m = 2n\) by \(N_{2n}\), we therefore have the recurrence relation \(N_{2n+4} = N_{2n+2} + N_{2n}\). Setting \(N_2 = 1\) and, to make the recurrence work, \(N_0 = 1\), one sees that we generate the well known Fibonacci sequence 1, 1, 2, 3, 5, 8, 13, \ldots. That is,

\[
N_{2m} = F_{m+1} = \frac{1}{\sqrt{5}}(\phi^m - (-1/\phi)^m).
\]

Here \(\phi = (1 + \sqrt{5})/2\) the so called golden ratio.

This formula for Fibonacci numbers was first written down by Euler in 1765, was forgotten, and rediscovered by Jacques Binet in 1843. For even moderate values of \(m\) one can obtain the correct result by just rounding the estimate \(\frac{1}{\sqrt{5}}\phi^m\). For example, for \(m = 9\) we have \(F_{10} = 55\) and \(\frac{1}{\sqrt{5}}\phi^{10} = 55.0036\).

Note that I have just quoted the result for the Fibonacci numbers - it is left as an exercise to look up one of any number of books for the derivation of this wonderful formula.

6. John is rowing upstream when his basketball cap blows into the water. Because he wore it backwards, he didn’t notice! He continues rowing for 10 minutes before he realises that he has lost his hat. He turns around (instantly) and rows downstream, catching up to his hat 1 km from where it
had fallen. What is the speed of the current?

Solution: There are two ways of doing this, depending on whether you know some Physics (the right sort of course) or not. The elegant solution, which requires familiarity with moving coordinate systems, requires you to consider the coordinate system attached to John’s hat. With respect to the hat, John rows for 10 mins. upstream, and so will need to row for 10 mins. downstream to catch his hat. His total rowing time is 20 min, during which time his hat has been swept downstream 1 km. Hence the speed of the current is 3 km/h.

Alternatively, and without such considerations, let the current flow at \( c \) km/min and let John’s rowing speed in still water be \( j \) km/min. John’s hat blows off. He rows upstream for 10 min., during which time he has travelled \( 10(j - c) \) km. upstream. The hat meanwhile has travelled \( 10c \) km. downstream. Now John turns around and rows for \( t \) mins until he catches his hat. After \( t \) mins he is \( t(j + c) - 10(j - c) \) km downstream, while his hat is \( 10c + tc \) km downstream. Equating these gives \( t = 10 \). The hat has drifted 1 km while John has been rowing for a time of 20 mins, giving, as above, a current speed of 3 km./h.

7. Chord \( AB \) divides a circle into two arcs with midpoints \( M \) and \( N \). A rotation about point \( A \) by some angle maps point \( B \) to \( B' \) and point \( M \) to \( M' \). Prove that the segments connecting the midpoint of segment \( BB' \) with points \( M' \) and \( N \) are perpendicular.

Solution: (Comment: Only one student successfully attempted this question.) Let \( O, P \) be midpoints of \( AB, BB' \) respectively. As \( MN \) is a diameter, so \( \angle MAN = 90^\circ \). Also \( \angle BON = 90^\circ \), as \( MN \) is the perpendicular bisector of \( AB \), and \( \angle AMN = \angle OBN \). (Equal angles on the same arc \( AN \).) Hence two of the angles of triangles \( MAN \) and \( BON \) are equal, so they are similar. Therefore

\[
\frac{NA}{AM} = \frac{NO}{OB} \tag{1}
\]

As \( O, P \) are mid-points of \( AB, BB' \) respectively,

\[
\triangle OBP \sim \triangle AB'B' \sim \triangle AM'M
\]

(by construction of \( B', M' \)). Therefore

\[
OB = OP, AM = AM' \tag{2}
\]

and

\[
\angle MAM' = \angle BOP. \tag{3}
\]

Therefore

\[
\frac{NA}{AM'} = \frac{NA}{AM} = \frac{NO}{OB} = \frac{NO}{OP} \tag{4}
\]

by (2), (1) and (2) respectively. Also,

\[
\angle NAM' = \angle NAM + \angle MAM' = \angle NOB + \angle BOP = \angle NOP \tag{5}
\]

where \( \angle MAM' \) is negative if \( M \) is rotated clockwise, and where the second equality follows from eqn. (3). Eqns. (4) and (5) imply that

\[
\triangle NAM' \sim \triangle NOP \tag{6}
\]

As these two triangles are of the same orientation, the angles between corresponding sides are equal. Therefore

\[
\angle M'NP = \angle ANO \tag{7}
\]

Also

\[
\frac{M'N}{NP} = \frac{AN}{NO} \tag{8}
\]

from (6). Now (7) and (8) imply that triangles \( M'NP \) and \( ANO \) are similar, so \( \angle M'PN = \angle AON = 90^\circ \). That is, \( M'P \perp PN \). Q.E.D.

There is also an alternative solution in terms of complex numbers, which also requires the knowledge that rotation about a point in the complex plane is represented by multiplication by a complex number of unit magnitude. This is left as an exercise!