1 At noon a train leaves Canberra for Sydney and another train leaves Sydney for Canberra. Being express trains, they maintain constant speed and make no stops. The first train requires four hours to complete the trip and the second train requires five hours. At what time do the trains meet? (You can ignore the initial acceleration of the trains, and assume that they travel with uniform speed throughout the journey.)

The distance apart doesn’t matter — it just scales the speed of the two trains. However, let’s call it $d$ km. (you could just as well have chosen a distance, such as 400 km.) Let one train travel with speed $h$ km/h, and the other with speed $w$ km/hr. Let them pass after $t$ hours. Thus $ht + wt = d$. But we are told that $5h = 4w = d$. Hence $w = 5h/4$, and so $9ht/4 = 5h$, or $t = 20/9$ hours, or $2\frac{2}{9}$ hours. Hence the trains met at 2:13:20 pm.

2 A triangle ABC has $AB = 6$, $BC = 8$ and $CA = 10$. Point D is the mid-point of AB, point E is on segment AC with $AE = 3$, and point F is on segment BC with $BF = 1$. Show that angle DEF can be determined and find its value.

The triangle is a right-angled triangle ($6^2 + 8^2 = 10^2$) with the angle at $B$ being a right angle. Observing that triangles ADE and CEF are isosceles, let $\angle AED = \angle ADE = x$ and $\angle CEF = \angle CFE = y$. Then as angles $DAE$ and $ECF$ sum to $90^\circ$ in $\triangle ABC$, $180 - 2x + 180 - 2y = 90$, from which $\angle DEF = 180 - x - y = 45$ degrees.
3 Show that there are no positive integers \( m \) and \( n \) such that \( m(m + 1) = n(n + 2) \).

If there are two such integers, either \( (a) \ m > n \) or \( m < n \). (By inspection, \( m \) cannot be equal to \( n \), as this implies \( 1 = 2 \).) If \( m < n \), then \( m(m + 1) < n(n + 2) \), as \( m < n \) and hence \( m + 1 < n + 2 \). So \( m \) cannot be less than \( n \). If \( m > n \), write \( m + k = n \), where \( k \geq 1 \). Substitution into the original equation gives \( m^2 + m = m^2 + m(2k + 1) + k(k + 1) \). Comparing coefficients of powers of \( m \) shows that there is no integral solution for \( k \) other than \( k = 0 \).

More elegantly: Adding 1 to both sides of the equation gives \( m^2 + m + 1 = n^2 + 2n + 1 = (n + 1)^2 \). But as \( m^2 < m^2 + m + 1 < m^2 + 2m + 1 = (m + 1)^2 \), \((n + 1)^2\) is between two consecutive perfect squares which is impossible. Hence the equation has no solutions.

4 I want to make a square lawn. I’ll use two sprinklers to water my lawn. Each sprinkler can irrigate a circular area of radius 5 metres. The water from the sprinklers must cover the entire lawn. I can place the sprinklers anywhere. How large can my lawn be?

Consider the pattern produced by two identical overlapping circles (of radius 5m.) By symmetry, the required square must have equal areas in both circles. Drawing some sketches suggests that the required square must have two sides parallel to the common chord of the two circles produced by joining their two points of intersection. The square can have side length no larger than the length of this chord. Clearly, the maximum length will occur when the circles are a distance apart such that the line perpendicular to the chord, passing through one of the points of intersection, and bounded by the perimeter of the circle, is the same length as the chord. Denote the chord length by \( 2a \). By Pythagoras, \( 5a^2/4 = 5^2 = 25 \). The square is therefore of size \( 4a^2 = 80 \text{sq. m.} \) (The other symmetric possibility is with the diagonal of the square coinciding with the common chord. This gives a much smaller area of 50 sq. m.)

5 Show that

\[
\frac{\sqrt{3} - \sqrt{5}}{\sqrt{2} + \sqrt{7} - 3\sqrt{3}}
\]

can be written in the form \( \sqrt{a}/b \), where \( a \) and \( b \) are relatively prime integers. (Show your working. Don’t merely give the answer).

Turning the equation upside down, so that

\[
\frac{\sqrt{2} + \sqrt{7} - 3\sqrt{5}}{\sqrt{3} - \sqrt{5}} = \sqrt{\frac{b}{a}}
\]

and rationalising the denominator by squaring both sides and multiplying both numer-
ator and denominator by $3 + \sqrt{5}$ gives
\[
(3 + \sqrt{5}) \frac{9 - 3\sqrt{5} + 2\sqrt{14 - 6\sqrt{5}}}{4} = \frac{b}{a}.
\]
Now note that $\sqrt{14 - 6\sqrt{5}} = 3 - \sqrt{5}$. The above equation thus simplifies to
\[
(3 + \sqrt{5})(3 - \sqrt{5})/4 = 5(9 - 5)/4 = 5 = \frac{b}{a}.
\]
Hence $b = 5$ and $a = 1$. (Note that one might “guess” that $\sqrt{14 - 6\sqrt{5}} = a - b\sqrt{5}$, then square both sides to give two equations for $a$ and $b$.)

6. A triangle has sides of length $a$, $b$ and $c$, which satisfy the equation
\[
1/(a + b) + 1/(a + c) = 3/(a + b + c).
\]
Determine the angle opposite the side of length $a$.

We rearrange the given equation as follows:
\[
1/(a + b) + 1/(a + c) = (2a + b + c)/(a + b)(a + c) = 3/(a + b + c).
\]
Cross multiplying the last equality and cancelling equal terms gives
\[
b^2 + c^2 = a^2 + bc.
\]
Now $2\cos \alpha = (b^2 + c^2 - a^2)/bc = 1r$, where the first equality follows from the cosine rule. Hence $\alpha = 60^\circ$. Only one student correctly solved this question.

7. Find the minimum value of
\[
\frac{(x + 1/x)^6 - (x^6 + 1/x^6) - 2}{(x + 1/x)^3 + (x^3 + 1/x^3)}
\]
for $x > 0$.

\[
\frac{(x + 1/x)^6 - (x^6 + 1/x^6) - 2}{(x + 1/x)^3 + (x^3 + 1/x^3)} = \frac{(x + 1/x)^6 - (x^6 + 1/x^6 - 2)}{(x + 1/x)^3 + (x^3 + 1/x^3)}
\]
\[
= \frac{(x + 1/x)^6 - (x^2 - 1/x^2)^2}{(x + 1/x)^3 + (x^3 + 1/x^3)}
\]
\[
= (x + 1/x)^3 - (x^3 + 1/x^3) \quad \text{difference of squares}
\]
\[
= 3(x + 1/x)
\]

As $x + 1/x = (\sqrt{x} - 1/\sqrt{x})^2 + 2 \geq 0 + 2 = 2$, the expression has the minimum value of 6, which occurs when $x = 1$. 

3