1. There are six TV programmes which run for the following times (in hours):

\[
\frac{3}{7} \quad \frac{2}{3} \quad \frac{3}{4} \quad \frac{19}{21} \quad 1\frac{7}{12} \quad 1\frac{2}{3}
\]

If two 3-hour videotapes are available, is it possible to store three of the programmes entirely on one tape and the other three entirely on the other tape?

**Solution.**

This requires finding three of the six given numbers which add to at most 3, such that the remaining three also add to at most 3. The sum of all the numbers is 6, so the constraints must be exactly satisfied. Since one of the numbers has 21 as its denominator, it is impossible to get a whole number total unless this is added to other fractions with denominators divisible by 7, and by 3. As \(\frac{19}{21} + \frac{3}{7} = \frac{4}{3}\), this leads to the solution

\[
\frac{19}{21} + \frac{3}{7} + 1\frac{2}{3} = 3, \quad \frac{2}{3} + \frac{3}{4} + 1\frac{7}{12} = 3.
\]

So the answer is yes.

2. As depicted in the diagram, points 1 cm along the edges of a square from its vertices create a smaller square, whose area is 14 sq cm less than the area of the larger square. Find the dimensions of the larger square.

**Solution.**

The difference in area of the two squares is the total area of the four triangles in the picture, so this is 14 sq cm. These triangles all have hypotenuse equal to a side of the smaller square, and all have the same angles, so they are all congruent to each other. So the area of one triangle is \(3\frac{1}{2}\) sq cm. As one of the sides is 1 cm, the other must be 7 cm. So the larger square has side length 8 cm.

3. Each of the 60 students (girls and boys) at an Easter camp found a number of eggs in the egg hunt. The girls found more, so each girl gave two eggs to the teachers. The teachers supplied each boy with one of these eggs and found that there were two left over for each teacher. If there were four times as many girls as teachers, how many girls were there?
Solution.
Let $x$ be the number of teachers. There were four times as many girls as teachers, so the number of girls was $4x$. There were sixty girls and boys altogether, so there were $60 - 4x$ boys. Each girl gave two eggs to the teachers, so there were $4x \times 2 = 8x$ eggs given to teachers. Each boy received one of these, so $8x - (60 - 4x)$ eggs remained for teachers. This was two for each teacher, so

$$8x - (60 - 4x) = 2x.$$ 

This is solved for $x$ as follows:

$$\begin{align*}
8x - 60 + 4x &= 2x \\
10x &= 60 \\
x &= 6.
\end{align*}$$ 

So there were $4x = 24$ girls.

4. Bridget has a rectangular brick of cement which is too large by 5cm in each dimension (length, breadth and height). She is about to trim the excess off, when Jill passes by and observes that she can get two identical bricks each of exactly the desired dimensions by splitting the brick in half. What are the dimensions of the brick?

Solution.
The first thing to notice is that the desired dimensions are all smaller than the original. If the original are $x$, $y$ and $z$ then the trimmed ones are $x-5$, $y-5$ and $z-5$ (all in cm). If the brick is split in half to achieve these dimensions, its longest side must be cut in half — otherwise the split brick would have one dimension equal to the longest. So, assuming $x \geq y \geq z$, the dimensions after splitting are $\frac{1}{2}x$, $y$ and $z$. These equal, in some order, $x-5$, $y-5$ and $z-5$. Which is $y$? Since $y-5 < y$ and $z-5 < y$, it must be that $x-5 = y$. Which is $z$? Since $z-5 < z$ we are only left with $y-5 = z$. This leaves only $z-5 = \frac{1}{2}x$. Finding everything in terms of $x$:

$$y = x - 5, \quad z = y - 5 = x - 10, \quad \frac{1}{2}x = z - 5 = x - 15.$$ 

Solving $\frac{1}{2}x = x - 15$ gives $15 = \frac{1}{2}x$ and so $x = 30$. Thus the dimensions of the original brick are 30, 25 and 20.
5. Find all prime numbers \( n \) such that \( n \) is both a difference of two primes and a sum of two primes. Note: 1 is not prime.

**Solution.**
If \( n \) is a difference of two primes, there are two cases. Firstly, both primes may be odd, in which case \( n \) is even and so \( n = 2 \) (this being the only even prime). But 2 is not a sum of two primes, so this is not the \( n \) we are looking for. We are left with the second case: one of the primes is even and is therefore equal to 2. Thus \( p - 2 = n \) for some prime \( p \). But then \( p \), and so also \( n \), must be odd, and so if \( n \) is a sum of two primes then those two primes cannot both be odd. So one of them is equal to 2, and \( n = 2 + q \) for some prime \( q \). Now we have three numbers, \( q = n - 2 \), \( n \), and \( p = n + 2 \), all prime and all odd. But for any three consecutive odd numbers, one of them must have a factor 3 (since multiples of 3 include every third odd number). The only prime with a factor 3 is 3 itself. Since 1 is not prime, the numbers are not 1, 3, 5, so the only other possibility is 3, 5, 7. Thus \( n = 5 \). (5 is the difference of the two primes 2 and 7, and the sum of 2 and 3.)

6. Aaron wrote down all the five-digit numbers which use only the digits 1, 2, 8 and 9, with repetitions permitted. (1024 numbers in all, starting with 11111.) Yuri added them all up (correctly) to find the total. Without writing down all the numbers, find Yuri’s total.

**Solution.**
It is helpful to notice first why there are 1024 five-digit numbers of this kind: there are four choices for the first digit, and for each of these alternatives there are four independent choices for the second, and so on, making \( 4 \times 4 \times 4 \times 4 \times 4 = 1024 \) choices for the five digits. Choosing “1” for the first digit leaves \( 4 \times 4 \times 4 \times 4 = 256 \) choices for the remaining digits, so this is how many of the numbers start with 1. For the same reason, the number of numbers starting with “2”, with “8” or with “9” is 256. These digits all have place value 10,000 in the five-digit number, so they contribute

\[
256 \times 10000 + 256 \times 20000 + 256 \times 80000 + 256 \times 90000 = 51200000
\]

to Yuri’s total. For similar reasons, 256 of the numbers have any given digit in the second position, with place value 1000, and so these digits contribute \( 256 \times 20 \times 1000 = 5120000 \) to Yuri’s total. Counting the same way for the digits in the third, fourth and fifth positions determines the contributions from those digits: 512000, 51200 and 5120. So Yuri’s
total is
\[
51200000 + 5120000 + 512000 + 51200 + 5120 = 512 \times 111110 = 56,888,320.
\]

7. Exactly 28 of the numbers 1, 2, \ldots, 30 (inclusive) are factors of a number \( N \). The other two numbers, which are not factors of \( N \), are both factors of a number \( M \) which is less than 250. What is \( M \)? (You do not have to find \( N \).)

Solution.
Let \( i \) and \( j \) be the two numbers missing from the list of factors, and suppose \( i < j \leq 30 \). An important thing to notice for this question is: if every number which is a prime power and is factor of \( x \) is also a factor of \( N \), then \( x \) itself is a factor of \( N \). Turning this around, if \( x \) is missing from the list of factors of \( N \), then one of its prime power factors must be missing. Using this fact, there are several ways to decide what \( i \) and \( j \) can be. For instance, many numbers can be eliminated from being \( i \) or \( j \) by an argument like the following. If 24 is missing, then one of its prime power factors must also be missing, so either 8 or 3 is missing. But if 3 is missing, so are 6, 9, etc, and if 8 is missing, so is 16. In either case there are at least three missing numbers less than 30; so 24 cannot be missing.

Here is an argument that takes a more general approach. The smallest number missing from the factors, \( i \), must be a prime power. Next, \( 2i \) and \( 3i \) cannot be factors of \( N \) because \( i \) is not a factor. So either \( i \leq 15 \), in which case \( j = 2i \) and \( 3i > 30 \), or \( i > 15 \). Taking the first case, \( i \leq 15 \) and \( 3i > 30 \) means \( i > 10 \) and so, since it’s a prime power, \( i = 11 \) or 13. Then \( j = 2i \) means \( i = 11 \), \( j = 22 \) is one possibility for the numbers, and \( M \) can be any multiple of 22 less than 250. (Note that if \( N \) is the product of the numbers from 1 to 30 excluding 11 and 22, then \( N \) has no factor 11, and so these numbers are indeed not factors of \( N \).) If \( i = 13 \) then \( j = 26 \) and \( M \) is any multiple of 26 less than 250. The second case is that \( i > 15 \). But then \( j \) must be a prime power too, since otherwise (from the argument above) one of its prime power factors must also be missing, and this would be less than 15. Since \( j \) is not a multiple of \( i \), \( j \) and \( i \) must be powers of different primes, and so the smallest \( M \) they both divide is \( i \times j \), which is too big (at least \( 16 \times 17 \)).

Hence the only solutions are the ones found in the first case: \( M \) is any multiple of 22 or 26 which is less than 250 (there are 11 such multiples of 22 and 9 such multiples of 26).