1. The symbol \( n! \) means the product \( n \times (n-1) \times (n-2) \times \ldots \times 2 \times 1 \). For example, \( 5! = 5 \times 4 \times 3 \times 2 \times 1 = 120 \). Find \( n \) such that \( n! = 2^{15} \times 3^6 \times 5^3 \times 7^2 \times 11 \times 13 \).

**Solution:** We look first for prime factors. Since \( n! \) has a factor of 13 and no factor of 17, we conclude that \( 13 \leq n < 17 \). Since 5 is a factor, so must 15 be (we only get factors of 5 from 5, 10, 15 etc.). Thus \( n = 15 \) or \( n = 16 \).

Let’s check the number of factors of 2 in 16! We get one factor of 2 from 2, 6, 10 and 14, we get two factors of 2 from 4 and 12 and three factors of 2 from 8 and four from 16, giving a total of 15 factors of 2, as found in \( n! \). Thus \( n = 16 \).

2. The symbol \( \lfloor x \rfloor \) means the greatest integer less than or equal to \( x \). Thus \( \lfloor 5.7 \rfloor = \lfloor 5.3 \rfloor = \lfloor 5 \rfloor = 5 \). Calculate the sum \( \lfloor \sqrt{1} \rfloor + \lfloor \sqrt{2} \rfloor + \lfloor \sqrt{3} \rfloor + \ldots + \lfloor \sqrt{48} \rfloor + \lfloor \sqrt{49} \rfloor + \lfloor \sqrt{50} \rfloor \).

**Solution:** For \( k \) a positive integer such that \( k^2 \leq n < (k+1)^2 \), so that \( k \leq \sqrt{n} < k+1 \) and so \( \lfloor \sqrt{n} \rfloor = k \). Thus for \( 1 \leq n \leq 3, \lfloor \sqrt{n} \rfloor = 1 \), and for \( 4 \leq n \leq 8, \lfloor \sqrt{n} \rfloor = 2 \), and for \( 9 \leq n \leq 15, \lfloor \sqrt{n} \rfloor = 3 \), etc. So the sum equals \( 3(1) + 5(2) + 7(3) + 9(4) + 11(5) + 13(6) + 2(7) = 217 \).

3. The sequence of numbers \( \ldots, a_{-3}, a_{-2}, a_{-1}, a_0, a_1, a_2, a_3, \ldots \) is defined by \( a_n = (n+1)a_{n-2} - (n+3)a_{n-1} \), for all integers \( n \). Calculate \( a_0 \).

**Solution:** From the recurrence, we observe that there are only two choices of \( n \) which result in equations containing \( a_2 \), \( n = 0 \) or \( n = 2 \). These give \( a_0 - a_2 = 9 \), and \( a_2 - 3a_0 = 25 \) respectively. Adding these together gives \(-2a_0 = 34 \) or \( a_0 = -17 \).

4. The triangle \( \triangle ABC \) is equilateral and the radius of its inscribed circle is 1. The line \( DE \) is drawn through \( C \), parallel to \( AB \), such that \( AEDB \) is a rectangle. A circle is drawn through the four vertices of the rectangle. What is its radius?

**Solution:** We first calculate the side length of the equilateral triangle \( ABC \). Let \( O \) be the centre of the smaller circle, and \( P \) the point of tangency of the circle to the side \( AB \). Join \( OP \) and \( OB \). Then \( \angle OBP = 90^\circ \) by tangency, and \( \angle OBP = 30^\circ \) by symmetry since \( \angle CBA = 60^\circ \). Since \( OP = 1 \) and \( \triangle BOP \) is a right-triangle, \( OB = 2, BP = \sqrt{3} \), and hence \( AB = 2\sqrt{3} \). Also, by symmetry \( CO = OB = 2 \), so \( CP = 3 \).

Now, since \( ABDE \) is a rectangle, \( AE = CP = 3 \), and \( BE \) is a diameter of the circumscribed circle. By Pythagoras, \( BE^2 = AE^2 + AB^2 = 21 \), so the diameter is \( \sqrt{21} \).
5. The octagon $P_1P_2P_3P_4P_5P_6P_7P_8$ is inscribed in a circle, with the vertices around the circumference in the given order. Given that the polygon $P_1P_3P_5P_7$ is a square of area 5 and the polygon $P_2P_4P_6P_8$ is a rectangle of area 4, find the maximum possible area of the octagon.

**Solution:** Here is an elegant solution, using complex numbers, due to D. J. Bernstein. The circle circumscribes a square of area 5, so the circle has radius $\sqrt{5}/2$. Hence the rectangle has sides $\sqrt{2}$ and $\sqrt{8}$. Without loss of generality, assume that $P_2P_4$ has length $\sqrt{2}$. Put $P_2$, $P_4$, $P_6$, $P_8$ into the complex plane at $\sqrt{2}(1/2 + i)$, $\sqrt{2}(-1/2 + i)$, $\sqrt{2}(1/2 - i)$. Put $P_1$ in the complex plane at $\sqrt{5}/2\exp(i\theta)$; then $P_3$, $P_5$, $P_7$ are at $i\sqrt{5}/2\exp(i\theta)$, $-\sqrt{5}/2\exp(i\theta)$, $-i\sqrt{5}/2\exp(i\theta)$. The triangles $P_8P_1P_2$ and $P_4P_5P_6$ each have area $\sqrt{5}\cos\theta - 1$. The triangles $P_2P_3P_4$ and $P_6P_7P_8$ each have area $\sqrt{5}/4\cos\theta - 1$. Hence the octagon has area $3\sqrt{5}\cos\theta$. The maximum possible area is achieved at $\theta = 0$, and is $3\sqrt{5}$.

6. Three distinct points with integer coordinates lie in the plane on a circle of radius $r > 0$. Show that two of these points are separated by a distance of at least $r^{1/3}$.

**Solution:** This solution is due to D. Rusin. We first use the standard result that a triangle with sides $a$, $b$, $c$ and circumscribing circle of radius $r$ has area $abc/4r$. If $a, b, c < r^{1/3}$ then the area is less than $1/4r$; but the area is at least $1/2$ since the points have integer coordinates. (Also, $r \geq \frac{1}{2}$.) Hence two of these points are separated by a distance of at least $r^{1/3}$.

7. A *king* in the game of chess is allowed to move only one square at a time, either up or down, left or right, or to one of the four neighbouring diagonal squares. In this problem we consider the more restricted case of a king that cannot move downwards. The only allowed steps are up, to the right, or diagonally up and to the right. Let $r(n+1)$ denote the number of possible paths from the bottom-left square to the top-right square of such a restricted king, on an $(n+1) \times (n+1)$ board. For example, $r(2) = 3$, corresponding to up-right, right-up, and a single diagonal step. Show that

$$r(n+1) = \sum_{l=0}^{n} \binom{n}{l} \binom{2n-l}{n},$$

and hence, or otherwise, that $r(n+1)$ is the coefficient of $x^n$ in the expansion of $(1 - 6x + x^2)^{-1/2}$.

**Solution:** This solution is due to P. Grossman. If the king makes no diagonal moves, then he makes $n$ horizontal and $n$ vertical moves, for a total of $2n$. Each diagonal move reduces the total number of moves by one. Therefore, if $l$ denotes the number of diagonal moves, the king makes $2n - l$ moves in total.
Of these $2n - l$ moves, $n$ must be either horizontal or diagonal, since these are the moves that take the king to the next file on the board. These can be chosen in ${2n-l \choose n}$ ways. Of these $n$ moves, $l$ are diagonal, and these can be chosen in ${n \choose l}$ ways. Therefore the total number of ways the king can move using $l$ diagonal moves is the product of these two terms, which must then be summed over all possible values of $l$ to give the required result.

This part of the solution is due to M. Hirschhorn. Construct the generating function

$$
\sum_{n \geq 0} r(n + 1)x^n = \sum_{n \geq 0} \sum_{l=0}^{n} \binom{n}{l} \binom{2n-l}{n} x^n
$$

$$
= \sum_{l \geq 0} \sum_{n \geq l} \binom{n}{l} \binom{2n-l}{n} x^n
$$

$$
= \sum_{l \geq 0} \sum_{m \geq 0} \binom{m+l}{l} \binom{2m+l}{m+l} x^{l+m}
$$

$$
= \sum_{l,m \geq 0} \binom{2m}{m} \binom{2m+l}{2m} x^{l+m}
$$

$$
= \sum_{m \geq 0} \binom{2m}{m} x^m (1 - x)^{-2m-1}
$$

$$
= (1 - x)^{-1} \sum_{m \geq 0} \binom{2m}{m} \left( \frac{x}{(1-x)^2} \right)^m
$$

$$
= (1 - x)^{-1} \left( 1 - \frac{4x}{(1-x)^2} \right)^{-\frac{1}{2}}
$$

$$
= [(1-x)^2 - 4x]^{-\frac{1}{2}} = (1 - 6x + x^2)^{-\frac{1}{2}}
$$