1. Three teams collected peaches, each collecting a whole number of baskets full. Team A’s baskets held 16 peaches, team B’s baskets held 12 and team C’s baskets held 10. The team with the most peaches collected 480, the next collected 400 and the next 320. Which team collected the most?

Solution.
The size of a team’s basket must be a factor of the number of peaches it collected. Note that 16 and 10 are factors of 480, as well as 400 and 320. But 12 is not a factor of 400 or 320. So team B must have collected 480. That is, team B collected the most.

2. Farmer Brown, farmer White and farmer Black had equal numbers of cows before two years of drought. A quarter of farmer Brown’s cows died in the first year, and a third of the remainder in the second. A third of Farmer White’s died in the first year, and a quarter of the remainder in the second. A third of Farmer Black’s, plus four more, died in the first year, and a quarter of the remainder died in the second year. Farmer Black then bought four more. Who then had the most cows (or were they equal)? Why?

Solution.
Let $x$ be the number of cows that each farmer had at the beginning. In the first year, a quarter of farmer Brown’s cows died: that is $\frac{1}{4}x$. The number left was $x - \frac{1}{4}x = \frac{3}{4}x$. Notice that the number left now is $\frac{3}{4}$ of the original. In the second year, a third of these died, so $\frac{2}{3}$ of them were left, or $\frac{2}{3} \times \frac{3}{4}x = \frac{1}{2}x$. Farmer White loses a third in the first year, leaving $\frac{2}{3}x$, and in the second year a quarter of these died, leaving $\frac{3}{4} \times \frac{2}{3}x = \frac{1}{2}x$. Farmer Black lost a third in the first year, leaving $\frac{2}{3}x$, then lost four, making $\frac{2}{3}x - 4$, and then a quarter of these died in the second year, leaving $\frac{3}{4}(\frac{2}{3}x - 4) = \frac{1}{2}x - 3$. After buying four more, he had $\frac{1}{2}x + 1$. So Farmer Black had more at the end.

[Note that solutions which start by assuming there was a given number of cows, such as 60, do not answer the problem completely. The solution given here shows that the answer does not depend on the number of cows.]

3. The telephone keypad has the digits 1 to 9 arranged as follows:

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1 2 3
4 5 6
7 8 9
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All I can remember of my Aunt’s 9-digit telephone number is that it consists of three sets of three non-zero digits in a row (up or down, forwards, backwards or diagonal, like 6,5,4 or 7,5,3) with the 3rd the same as the 4th and the 6th the same as the 7th. I spend 40 cents and try 369-963-357, but it is the wrong number. I have $45 left. Is that enough to try all the other possible numbers?

Solution.
How many combinations need to be tried? If we start a three-digit row from a corner (1, 3, 7 or 9), there are three other corners to go to. For each of these, the next row has three ways to go, making $3 \times 3 = 9$ 6-digit combinations. The
final row again has three ways to go, making $9 \times 3 = 27$ numbers starting from a given corner. There are four corners, so $27 \times 4 = 108$ numbers start from corners. Finally, starting at 2, 4, 6 or 8 there is only one way to go, making four more numbers. Of the 112 numbers, I have already dialled one, so need to try another 111, costing $44.40. Yes, I do have enough money.

4. There is a pattern in the following list, but one number is missing. Find the value of the question mark:

$$9 \quad 64 \quad ? \quad 1296 \quad 3125 \quad 4096 \quad 2187 \quad 256 \quad 1.$$  

Solution.

It is easy to see that some of the numbers in this list are high powers of integers, like $64 = 2^6 = 4^3 = 8^2$. Looking at the prime factors does not give the best clue, but it helps to notice that all numbers are powers of integers less than 10: the best clues are $3125 = 5^5$ and $2187 = 3^7$. With a little luck one then sees the pattern: the numbers are

$$9^1 \quad 8^2 \quad ? \quad 6^4 \quad 5^5 \quad 4^6 \quad 3^7 \quad 2^8 \quad 1^9$$

and the question mark stands for $7^3 = 343$.

5. Four coins are flipped at random and then placed at the corners of a square. What are the chances (probability) that each of the four sides of the square has both a head and a tail? Next, repeat the question, with three coins at the corners of a triangle. Generalize to $n$ coins placed at the corners of an $n$-sided polygon.

Solution.

There are $2 \times 2 \times 2 \times 2 = 2^4 = 16$ combinations of flipped coins, and each has the same chance of arising. Which ones satisfy the required condition? If H is at the first (say, top left) corner then T must be at the next, H at the next and T at the next: $\text{HT}$ $\text{TH}$. If T is first then H must be next, then T then H: $\text{TH}$ $\text{HT}$. These two are the only possibilities, so the chance that one of them occurs is 2 out of 16, or one out of 8. As a probability, we just express this as $1/8$.

For a triangle, there are no ways to satisfy the condition, since starting at a head on one corner, we must have a tail on the next, then a head on the next, which together with the first head produces a side with two heads. So for a triangle, the probability is 0.

For an $n$-sided polygon, the same arguments work: when $n$ is even, there are two valid combinations out of $2^n$, so the probability is $2/2^n$ or $1/2^{n-1}$. When $n$ is odd, the probability is 0 because there are no valid configurations of heads and tails: every second corner must have a head and every other a tail, so the total number of corners must be even.

6. On Monday I walk up an upward-moving escalator at one step per second, and it takes 60 seconds to reach the top. On Tuesday, it is moving at the same rate, and I take two steps per second, reaching the top in 36 seconds. On Wednesday, the escalator is stopped, so I run up at 3 steps per second. How long does it take?

Solution.

In walking up at 2 steps per second, I cover 48 steps, whilst at 1 per second I
cover 60 steps (taking 24 seconds longer). So 12 steps disappear into the top of the escalator every 24 seconds. Thus, in the 60 seconds required to cover it at one step per second, 30 steps disappear into the top of the escalator. Added to the 60 steps covered, this means the escalator is 90 steps long. So on Wednesday, it takes 30 seconds at the rate of 3 per second.

7. A quadrilateral ABCD has two parallel sides AB and CD. Joining the midpoints of the four sides of ABCD, we can form another quadrilateral PQRS. Explain why the area of PQRS is exactly half that of ABCD.

Solution.
First draw a perpendiculars YZ to DC through Q meeting the extension of BA at Y and meeting CD at Z.

Then the triangles SZD and SYA are congruent, because the angles at S are equal, they both have a right angle, and the length of SA equals the length of SD (since S is the midpoint of AD). Thus the length of SZ equals the length of SY. For the same reasons, Q is the midpoint of the perpendicular WX, and triangle BXQ is congruent to CWQ. Thus the rectangle WXYZ has the same area as the original quadrilateral ABCD.

We also know that YS has the same length as SQ (being half of YZ or XW) so YSQX is a rectangle. The formula for the area of a rectangle is base×height, and for a triangle it is \( \frac{1}{2} \times \text{base} \times \text{height} \). The triangle PSQ has the same base as YSQX and the same height, so its area is half. Similarly, triangle SRQ has half the area of SZWQ. Putting these two triangles together, we obtain that PQRS has half the area of WXYZ (and hence also half of ABCD).

Note: It is not enough to say that one can fold the four triangles outside PQRS in to cover PQRS. Folding them, they usually overlap each other, and leave some places uncovered. This does not work even starting with a non-square rectangle ABCD. Try it with a piece of paper!