1. A fruit shop sold red apples at four for a dollar, and green apples at three for a dollar. Another fruit shop sold red apples at the same price and green apples at six for a dollar. If you bought \( m \) red apples and \( n \) green apples from each store, and spent a total of \( \$8 \), how many apples did you buy?

**Solution:**
At the first shop: Since red apples are sold at four for a dollar, they cost \( \frac{1}{4} \) each and so you spend \( \frac{m}{4} \) on red apples. Since green apples are sold at three for a dollar, they cost \( \frac{1}{3} \) each and so you spend \( \frac{n}{3} \) on green apples.

At the second shop: Since red apples are sold at four for a dollar, they cost \( \frac{1}{4} \) each and so you spend \( \frac{m}{4} \) on red apples. Since green apples are sold at six for a dollar, they cost \( \frac{1}{6} \) each and so you spend \( \frac{n}{6} \) on green apples.

So you spend \( \frac{m}{4} + \frac{n}{3} + \frac{m}{4} + \frac{n}{6} = \frac{m+n}{2} \) overall on apples. Since this also equals \( \$8 \),

\[
\frac{m+n}{2} = 8
\]

\[
\Rightarrow m + n = 16
\]

\[
\Rightarrow 2(m + n) = 32
\]

But \( 2(m + n) \) is the total number of apples bought and so you must have bought 32 apples.

*Comment:* Many students assumed that an integral number of dollars was spent on each type of apple at each shop. The above solution shows this need not be the case. In fact, provided \( m \) and \( n \) are any positive real numbers whose sum is 16, the answer remains the same, even though this may not be feasible in real life!

2. The 6 digits of Maja’s phone number are all different. If the product of the 6 digits ends in 4 what is the sum of the 6 digits if the sum is even?

**Solution 1:** 0 cannot be one of the digits because the product of the 6 digits would then end in 0. Since the product ends in 4, one of the digits must be even. Then 5 cannot be one of the digits since the inclusion of 5 and this even number means that the product of the 6 digits ends in 0.

Hence the 8 possible digits are 1, 2, 3, 4, 6, 7, 8 or 9. The sum of the 6 digits is even, so there must be an even number of odd digits. Since there are 4 even digits and 4 odd digits, there must either be 2 even digits and 4 odd digits OR 4 even digits and 2 odd digits.

If there are four odd digits, they must be 1, 3, 7 and 9. The two even digits must be chosen from 2, 4, 6 and 8. Considering the product of the 6 possible pairs of
even digits (2,4), (2,6), (2,8), (4,6), (4,8), (6,8) with $1 \times 3 \times 7 \times 9$, it can be verified that only $1 \times 3 \times 7 \times 9 \times 2 \times 8 = 3204$ ends in 4.

If there are four even digits, they must be 2, 4, 6 and 8. The two odd digits must be chosen from 1, 3, 7 and 9. Considering the product of the 6 possible pairs of odd digits (1,3), (1,7), (1,9), (3,7), (3,9), (7,9) with $2 \times 4 \times 6 \times 8$, it can be verified that only $2 \times 4 \times 6 \times 8 \times 3 \times 7 = 8064$ ends in 4.

Hence the 6 digits are either 1,3,7,9,2,8 or 2,4,6,8,3,7. In either case,

$$1 + 3 + 7 + 9 + 2 + 8 = 30 \quad \text{and} \quad 2 + 4 + 6 + 8 + 3 + 7 = 30$$

and so the sum of the 6 digits in Maja’s number is 30.

Solution 2: (Provided by Matthew Ng, Scotch College)

As in the previous solution, 0 and 5 cannot be digits.

Thus the digits come from 1,2,3,4,6,7,8,9. The last digit of the product of these is 6 so the product of the two digits to be taken away ends in 4 or 9. The sum of the digits to be taken away is even since $1+2+3+4+6+7+8+9$ is even hence we take away two odd digits or two even digits. The only two even digits with product ending in 4 is 4 and 6 and the only two odd digits with product ending in 9 is 1 and 9.

Since $4 + 6 = 10 = 1 + 9$, the sum of the 6 digits of Maja’s phone number is

$$1 + 2 + 3 + 4 + 6 + 7 + 8 + 9 - 10 = 30.$$

3. Place a square inside a circle and a circle inside the square and so on as in the diagram. How many squares do you have to place before the next circle is smaller than one third the size of the original circle?

Solution: Let $OB = R$ be the radius of the larger circle and $OM = r$ be the radius of the smaller circle.
In $\triangle MBO$, by Pythagoras’ Theorem,

\[
OM^2 + MB^2 = OB^2
\]
\[
\Rightarrow 2r^2 = R^2 \quad \text{since} \quad OM = MB = r
\]
\[
\Rightarrow \sqrt{2}r = R
\]
\[
\Rightarrow \frac{r}{\pi} = \frac{1}{\sqrt{2}}
\]

So the radius of the smaller circle is $\frac{1}{\sqrt{2}}$ the radius of the larger circle. We interpret ‘size’ to mean the radius (or diameter) of the circle. Then if $n$ squares are placed, the radius of the $(n + 1)^{th}$ circle will be $\left(\frac{1}{\sqrt{2}}\right)^n$ the radius of the original circle.

It suffices to find the smallest value of $n$ so that

\[
\left(\frac{1}{\sqrt{2}}\right)^n < \frac{1}{3}.
\]

It can be verified that $\left(\frac{1}{\sqrt{2}}\right)^3 > \frac{1}{3}$ and $\left(\frac{1}{\sqrt{2}}\right)^4 < \frac{1}{3}$. Hence 4 squares need to be placed.

Comment: Many students interpreted the ‘size’ to mean the area of a circle. The solution for this case is as follows.

The ratio of the areas of two successive circles equals the square of the ratio of the radii $\left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2}$. This follows from the fact that the area of a circle is proportional to the square of its radius. If $n$ squares are placed, the area of the $(n + 1)^{th}$ circle will be $\left(\frac{1}{\sqrt{2}}\right)^n$ the area of the original circle.

It suffices to find the smallest value of $n$ so that

\[
\left(\frac{1}{2}\right)^n < \frac{1}{3}.
\]

It can be verified that $\left(\frac{1}{2}\right) > \frac{1}{3}$ and $\left(\frac{1}{2}\right)^2 < \frac{1}{3}$. Hence 2 squares need to be placed.
4. A concrete square is surrounded by a grass border of constant width and the distance from one corner of the concrete to a corner of the grass border as shown in the diagram is 10 metres. If the area of the concrete square is $a$ and the area of the grass border is $b$, find $2a + b$.

Solution 1: Let the width of the concrete strip be $x$ metres and the side length of the square of grass be $y$ metres, as marked in the diagram. Then, by Pythagoras’ theorem on the right angled-triangle $ABC$,

$$x^2 + (x + y)^2 = 10^2 = 100$$

$$2x^2 + 2xy + y^2 = 100$$

Now we are asked to find $2a + b$. The total area of the grass and concrete is $a + b$, which must be equal to $(y + 2x)^2$. $a$ must also be equal to $y^2$, so we have

$$2a + b = a + b + a = (y + 2x)^2 + y^2 = 2(2x^2 + 2xy + y^2) = 200$$

Using the relation that we obtained earlier from Pythagoras’ theorem. So $2a + b = 200$ square metres.

Solution 2: (Provided by Navin Ranasinghe, Scotch College) Divide the diagram as shown on the left.
If we take these pieces, along with another square of the same size as the grass square, then we can rearrange the pieces into two squares like the one shown on the right. Note that, with $x$ and $y$ defined as in the first solution, the side length of the centre square on the right is equal to the length of the longer leg of the right-angled triangle minus the length of the shorter leg, which is $x + y - x = y$. So we have taken pieces of total area $2a + b$ and rearranged them into two squares with a side length of 10 metres. Hence, $2a + b = 200$ square metres.

**Solution 3:** Consider the right-angled triangle shown in the diagram.

By Pythagoras’ theorem, $x^2 + y^2 = 10^2 = 100$. Now $x$ is equal to half the length of the diagonal of the square of grass, so the area of $a$ is equal to $2x^2$. $y$ is equal to half the length of the diagonal of the total square, so the area of $a + b$ is equal to $2y^2$. Hence,

$$2a + b = a + (a + b) = 2x^2 + 2y^2 = 200$$

5. Find all positive integer solutions of $m^2 + 49^2 = n^2$.

**Solution:** Since $n$ and $m$ are positive integers, then $n > m$ and so $n - m > 0$.

$$m^2 + 49^2 = n^2$$

$$\Rightarrow \quad n^2 - m^2 = 49^2$$

$$\Rightarrow \quad (n - m)(n + m) = 7^4$$
Since $m$ and $n$ are positive integers, $(n-m)$ and $(n+m)$ are both positive factors of $7^4$ and their product equals $7^4$. The factors of $7^4$ are $7^0 = 1, 7^1 = 7, 7^2 = 49, 7^3 = 343$ and $7^4 = 2401$. So $(n + m)(n - m)$ must be $1 \times 2401$ or $7 \times 343$ or $49 \times 49$.

Since $m$ is a positive integer, then $n + m > n - m$. Then $n + m = 2401$ and $n - m = 1$ OR $n + m = 343$ and $n - m = 7$.

In the first case, $2n = (n + m) + (n - m) = 2401 + 1 = 2402$ and so $n = 1201$. Also $2m = (n + m) - (n - m) = 2401 - 1 = 2400$ and so $m = 1200$.

In the second case, $2n = (n + m) + (n - m) = 343 + 7 = 350$ and so $n = 175$. Also $2m = (n + m) - (n - m) = 343 - 7 = 336$ and so $n = 168$.

Hence the two solutions are $(m, n) = (1200, 1201)$ and $(m, n) = (168, 175)$.

6. Find the ratio of the areas of the two regular hexagons in the diagram.

![Diagram](image)

**Solution 1:** We can divide the diagram as shown. The triangles marked $x$ are all clearly congruent (they are equilateral and have the same side length) and hence have the same area, $x$. The triangles marked $y$ are also all congruent due to the symmetry of the diagram, and hence have the same area, $y$.

Now consider the $x$ and $y$ triangles that are shaded. They have equal bases, as marked, and also have equal heights. Therefore, they have the same area, so $x = y$. The area of the small hexagon is $6x$, and that of the larger hexagon is $12x + 6y = 18x$. So the ratio of the areas is $6x : 18x = 1 : 3$. 
Solution 2: Let $h$ be the height of the small hexagon, and $H$ the height of the large hexagon. We can then mark lengths as shown in the following diagram:

By Pythagoras’ theorem, $H^2 + h^2 = (2h)^2$, so $H^2 = 3h^2$. Now the ratio of the areas of two similar figures is equal to the square of the ratio of the lengths of the figures. So the ratio of the area of the smaller hexagon to the area of the larger is $h^2 : H^2 = h^2 : 3h^2 = 1 : 3$.

7. How many different triangles with integer centimetre side lengths and perimeter 2004 cm are there?

Solution: Remember that the triangle inequality says that, for $(a, b, c)$ to form the sides of a triangle, we must have $a + b > c$, $b + c > a$, $c + a > b$. First, we shall count the number of isosceles triangles. An isosceles triangle will have side lengths $(a, a, b)$ where $2a + b = 2004$ and $2a > b$. So $2a > 2004 \div 2 = 1002$, and $a > 501$.

So the legal isosceles triangles have side lengths

$(502, 502, 1000), (503, 503, 998), \ldots, (1001, 1001, 2)$.

There are, therefore, 500 isosceles triangles (including one equilateral triangle with side length $2004 \div 3 = 668$).

Now, let us count the number of ordered triples $(a, b, c)$ that satisfy $a + b + c = 2004$, $a + b > c$, $b + c > a$, $c + a > b$. This is equivalent to counting the number of ordered pairs of positive integers $(a, b)$ such that $a + b > 2004 - a - b$, $b + (2004 - a - b) > a$, $(2004 - a - b) + a > b$ (because $c = 2004 - a - b$). These can be simplified to: $a + b > 1002, 1002 > a, 1002 > b$. We can plot this on a graph:
We need to count the number of points in the shaded region. This is clearly equal to $1 + 2 + 3 + \ldots + 1000 = \frac{1000 \times 1001}{2} = 500500$.

In counting these triples, we have counted three different types of triangles: scalene triangles ($S$ of them), non-equilateral isosceles triangles ($I$ of them), and the single equilateral triangle. We will count a scalene triangle $(a, b, c)$ six times, as $(a, b, c), (a, c, b), (b, c, a), (b, a, c), (c, a, b)$ and $(c, b, a)$. We will count a non-equilateral scalene triangle $(a, a, b)$ three times, as $(a, a, b), (a, b, a)$ and $(b, a, a)$. We will count the equilateral triangle once. Hence, $6S + 3I + 1 = 500500$. We also know that $I + 1 = 500$, so $I = 499$. We can use this to find $S$:

$$6S = 500500 - 1 - 3 \times 499 = 499002$$

$$S = 83167$$

Hence, the total number of triangles is $S + I + 1 = 83167 + 499 + 1 = 83667$.

8. Given any three integers take the nine products of any two of these integers (so some numbers are repeated.) Show that there are 5 of these products whose sum is divisible by 5.

Solution: First, note that we need only consider the numbers $a, b, c$ modulo 5. All congruences in this solution will be taken modulo 5. Now, we have four cases:

Case 1. One of $a, b, c$ is congruent to 0 mod 5. Without loss of generality, let $a \equiv 0 \pmod{5}$. Then $a^2 + ab + ba + ac + ca \equiv 0 \pmod{5}$, so this sum of five of the nine numbers is a multiple of 5: we are done.
Case 2. $a \equiv b \equiv c \neq 0 \pmod{5}$. In this case, the nine numbers are all congruent to $a^2 \pmod{5}$, so the sum of any five of them will be equal to $5a^2$, which is clearly a multiple of 5: we are done.

Case 3. Exactly two of $a, b, c$ are congruent modulo 5, and none are congruent to 0 $\pmod{5}$. Without loss of generality, let $a \equiv b \pmod{5}$. Our nine numbers are then $a^2, a^2, a^2, ac, ac, ac, c^2$. Now, because $a \neq c \pmod{5}$, and neither is congruent to 0 $\pmod{5}$, $a^2 - ac \neq 0 \pmod{5}$. So, because 5 is a prime number, the numbers 0, $a^2-ac, 2(a^2-ac), 3(a^2-ac), 4(a^2-ac)$ are all different modulo 5. Therefore, they must include each possible remainder modulo 5 once each. Therefore, the numbers $(c^2+4ac)+0, (c^2+4ac)+a^2-ac, (c^2+4ac)+2(a^2-ac), (c^2+4ac)+3(a^2-ac), (c^2+4ac)+4(a^2-ac)$ must also include each remainder modulo 5 once each, including 0. So one of the numbers $c^2+4ac, c^2+3ac+a^2, c^2+2ac+2a^2, c^2+ac+3a^2, c^2+4a^2$ must be congruent to 0 modulo 5. These are all sums of five of our nine numbers, so that finishes this case.

Case 4. $a, b, c$ are all distinct modulo 5, and none are congruent to 0 $\pmod{5}$. $a, b, c$ must give three of the four remainders 1, 2, 3, 4 modulo 5. Let $k$ be the inverse of $d$ modulo 5 (this exists as 5 is a prime). Then the numbers $ka, kb, kc, kd$ must give all of the non-zero congruences modulo 5, and $kd \equiv 1 \pmod{5}$, so $ka, kb, kc$ must have remainders 2, 3, 4 in some order, modulo 5. Assume, without loss of generality, that $ka \equiv 2, kb \equiv 3, kc \equiv 4 \pmod{5}$. Now note that

$$k^2(a^2 + ab + ba + c^2 + ac) \equiv (ak)^2 + (ak)(bk) + (bk)(ak) + (ck)^2 + (ak)(ck)$$

$$\equiv 2^2 + 2 \cdot 3 + 3 \cdot 2 + 4^2 + 4 \cdot 2 \equiv 40 \equiv 0 \pmod{5}$$

(all modulo 5) Because $k^2 \neq 0 \pmod{5}$, we must have $a^2 + ab + ba + c^2 + ac \equiv 0 \pmod{5}$. This completes this case.