(1) Two shortsighted ladies who are friends live in two separate towns, joined by a road between the towns. They decide to visit each other, leaving their respective towns at exactly the same time one morning, and walking at constant speed towards the town of the other. One lady walks faster than the other, and they pass each other (without noticing) at mid-day of the same day. The quicker walker arrives at her destination at 4pm, the slower at 9pm. At what time did they set out?

(2) If $i$ and $j$ are integers such that $3i - 2j = 4$, what value or values can $2i - 3j$ take?

(3) Two pins or nails are inserted into a wall, and the corner of a rectangular sheet of cardboard is pushed between them, as in the figure below. As the cardboard is rocked back and forth, always keeping it in contact with the pins, what is the shape of the curve traced by the corner between the pins?

(4) Let $a$ and $b$ be positive integers. Prove that

$$\frac{a!b!}{a^b b^a} > \frac{(a + b)!}{(a + b)^{a+b}}.$$ 

(5) Let $m + 1$ be divisible by 24. Prove that the sum of all divisors of $m$ is also divisible by 24. (For example, 96 is divisible by 24, so 95 is one possible value for $m$. Its divisors are 1, 5, 19, and 95, which sum to 120, which is divisible by 24).

(6) Show that

$$2^p \times \cos \left( \frac{\pi}{2p+1} \right) \times \cos \left( \frac{2\pi}{2p+1} \right) \times \cos \left( \frac{3\pi}{2p+1} \right) \times \cdots \times \cos \left( \frac{p\pi}{2p+1} \right) = 1,$$

where $p$ is any positive integer.

(7) Assume you live in town $A$, and you have to walk to town $B$ for some unpleasant purpose. You therefore want to maximise the distance you walk. Assume further that you walk with constant speed, and that the towns are 10 km apart. Your aim is to walk from $A$ to $B$ in a sequence of $n$ straight line paths (not necessarily of the same length). Your distance from your destination must never increase (remember, you are walking towards $B$, not away from it).

First calculate the maximum distance that you can walk for the special cases $n = 2$ and $n = 3$. Then find the maximum distance in the general case — your answer will, of course, be a function of $n$. 
1. Suppose that the ladies set out $t$ hours before midday. Let the speed of the faster lady be $u$, and the speed of the slower lady be $v$. Then the faster lady took $t$ hours to cover the distance that the slower lady took 9 hours to cover, so letting this distance be $m$, we have:

$$\frac{u}{v} = \frac{\frac{m}{9}}{t} = \frac{9}{t}.$$ 

Similarly, the faster lady took 4 hours to cover the distance that the slower lady took $t$ hours to cover, so letting this distance be $n$, we have:

$$\frac{u}{v} = \frac{n}{\frac{t}{4}} = \frac{t}{4}.$$ 

Hence, equating the two expressions for $\frac{u}{v}$, we have:

$$\frac{t}{4} = \frac{9}{t} \quad \text{and} \quad t^2 = 36 \quad \text{so} \quad t = \pm 6.$$ 

We know that the ladies set off before midday, so $t > 0$. Hence, $t = 6$, so the ladies set off at 6am.

2. Rearranging the given equation, we get $3i = 2(j + 2)$. This tells us that $3i$ is even, and so $i$ must be even (since 3 is odd). So we can write $i = 2k$, where $k$ is an integer. Substituting this into the equation and simplifying gives,

$$6k = 2(j + 2) \quad \text{so} \quad \begin{align*}
6k &= 2(j + 2) \\
3k &= j + 2 \\
j &= 3k - 2.
\end{align*}$$ 

We now have both $i$ and $j$ in terms of $k$. Substituting this into the expression of interest,

$$2i - 3j = 2(2k) - 3(3k - 2) = -5k + 6.$$ 

As $k$ varies, this expression can only take values in the set $\{\ldots, -9, -4, 1, 6, \ldots\}$, i.e. all integers that are 1 more than a multiple of 5. In fact, all such values are possible since,
given any \( k \), letting \( i = 2k \) and \( j = 3k - 2 \) ensures that \( 3i - 2j = 4 \) while making \( 2i - 3j = -5k + 6 \).

Thus, \( 2i - 3j \) can take any value that is 1 more than a multiple of 5.

3. The shape will be a semicircle. To prove this, let the nails be labelled \( A \) and \( B \), with the point halfway between the nails labelled \( M \). Now suppose that the corner of the cardboard is at a position \( C \). Then, because the corner of the cardboard is a right angle, \( \angle ACB = 90^\circ \).

We will show that \( CM = AM = BM \). To do this, let \( D \) be the point obtained by rotating point \( C \) about \( M \) by \( 180^\circ \) (see diagram below). Then \( M \) is the midpoint of \( AB \) and of \( CD \), so \( ACBD \) must be a parallelogram. Furthermore, \( \angle ACB = 90^\circ \), so \( ACBD \) must be a rectangle. The lengths of the diagonals of a rectangle are equal, so \( AB = CD \). Hence \( AM = \frac{AB}{2} = \frac{CD}{2} = CM \) as required.

Hence, no matter how the cardboard is ‘rocked’, the distance between the corner and \( M \) is always a constant, so the corner moves on a semicircle. It is clear that the cardboard can be rocked so that the corner lies anywhere on the semicircle, so the shape traced by the corner is a semicircle.

4. Using the binomial expansion formula to expand \( (a + b)^{a+b} \), we get,

\[
(a + b)^{a+b} = \binom{a+b}{0} a^0 b^{a+b} + \cdots + \binom{a+b}{i} a^i b^{a+b-i} + \cdots + \binom{a+b}{a} a^a b^{b}.
\]

Since \( a \) and \( b \) are positive, each term on the right hand side is positive. Therefore, if we select out only one term, this will be smaller than the left hand side. In particular, selecting the \( b \)th term gives,

\[
(a + b)^{a+b} > \binom{a+b}{b} a^{b} b^{a}.
\]

The binomial coefficient can be written in terms of factorials,

\[
\binom{a+b}{b} = \frac{(a + b)!}{a! \ b!}.
\]
Substituting this into the above inequality and rearranging gives us the desired result,

\[
(a + b)^{a+b} > \frac{(a + b)!}{a! b!} \cdot \frac{a^b b^a}{a+b}.
\]

5. After doing a few examples, it becomes clear that the factors of \( m \) ‘pair up’ into pairs which add to a multiple of 24. For example, \( 95 = 1 \times 95 \), and \( 1 + 95 = 96 \), which is a multiple of 24. Also, \( 95 = 5 \times 19 \), and \( 5 + 19 = 24 \), which is a multiple of 24. So \( 1 + 5 + 19 + 95 = 96 + 24 \) which is a sum of multiples of 24, so it must itself be a multiple of 24. We just want to prove that this always happens.

Suppose that \( a \) and \( b \) are integers and \( ab = m \). We want to show that \( a + b \) is a multiple of 24.

Now, as \( m \) is one less than a multiple of 24, \( \gcd(m, 24) = 1 \). Hence, both \( a \) and \( b \) do not have a common factor with 24. Now we show that for any number \( x \) that has no common factor with 24, \( x^2 - 1 \) is divisible by 24.

As \( x \) has no common factor with 24, it is odd. Thus \( x = 2k + 1 \), and

\[ x^2 - 1 = (2k + 1)^2 - 1 = 4k(k + 1). \]

Now one of \( k, k + 1 \) must be even, so \( k(k + 1) \) is even. Hence \( 4k(k + 1) \) is divisible by 8, so \( x^2 - 1 \) is divisible by 8.

As \( x \) has no common factor with 3, either \( x + 1 \) or \( x - 1 \) is divisible by 3. Hence \( x^2 - 1 = (x + 1)(x - 1) \) is divisible by 3. So \( x^2 - 1 \) is divisible by 3 and 8, hence it is divisible by 24.

Now, let \( a + b = p \). Now we know that

\[ m + 1 = ab + 1 = a(p - a) + 1 = ap - (a^2 - 1). \]

We know that \( m + 1 \) is divisible by 24, and we showed above that \( a^2 - 1 \) is divisible by 24, so \( ap = (m + 1) + (a^2 - 1) \) must also be a multiple of 24. But \( a \) does not have a common factor with 24, so \( p \) is a multiple of 24, which is what we wanted to show originally.

Hence,

\[
\sum_{a|m} a = \sum_{a^2 < m, a|m} (a + \frac{m}{a})
\]

(here, the notation \( a|m \) means ‘\( a \) divides \( m \)’; \( \sum_{a|m} \) means the sum over all numbers \( a \) which divide \( m \), and \( \sum_{a^2 < m, a|m} \) means the sum over all numbers \( a \) whose square is less than \( m \) and which divide \( m \)). We have shown above that, when \( a \) divides \( m \), \( a + \frac{m}{a} \) is a multiple of 24. Thus, the right hand side is a sum of multiples of 24, and is hence a multiple of 24.
There is one small problem with this argument. What if there is a divisor $a$ of $m$ such that $a \times a = m$? If this happens, then the $a$ term only gets added once, not twice, and so we won’t get our multiple of 24. But $m$ is one less than a multiple of 24, so it is odd. Thus $a$ would have to be odd. But the square of an odd number is always 1 more than a multiple of 8, as shown above. So $m = a^2$ would simultaneously be 1 more than a multiple of 8, and 1 less than a multiple of 24, which would mean that both $m - 1$ and $m + 1$ would be multiples of 8, which clearly can’t occur.

**Alternative Solution:** This solution uses some modular arithmetic to prove that, if $ab = m$, then $a + b$ is a multiple of 24.

As $m$ is 1 less than a multiple of 24, $m \equiv -1 \pmod{24}$. Hence, $a \times (-b) \equiv 1 \pmod{24}$. Thus $-b$ is the inverse of $a$ modulo 24. As $a$ is relatively prime to 24, it can only be congruent to 1, 5, 7, 11, 13, 17, 19 modulo 24. Note that the square of each of these numbers is congruent to 1 modulo 24, so each number is self inverse. Hence, $-b \equiv a \pmod{24}$, as $-b$ is the inverse of $a$ modulo 24, and $a$ is its own inverse. Thus, $a + b \equiv 0 \pmod{24}$, as required. The rest of the proof proceeds as above.

6. (Based on the partial solution provided by Muhammad Adib) Let

$$\cos\left(\frac{\pi}{2p+1}\right)\cos\left(\frac{2\pi}{2p+1}\right)\ldots\cos\left(\frac{p\pi}{2p+1}\right) = A.$$ 

Now, one of the double angle formulae is: $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$. How can we use this? We multiply $A$ by another quantity so that we can use this formula:

$$\sin\left(\frac{\pi}{2p+1}\right)\sin\left(\frac{2\pi}{2p+1}\right)\ldots\sin\left(\frac{p\pi}{2p+1}\right) = 2\sin\left(\frac{\pi}{2p+1}\right)\cos\left(\frac{2\pi}{2p+1}\right)\times 2\sin\left(\frac{2\pi}{2p+1}\right)\cos\left(\frac{3\pi}{2p+1}\right)\times\ldots\times 2\sin\left(\frac{p\pi}{2p+1}\right)\cos\left(\frac{p\pi}{2p+1}\right)$$

$$= \sin\left(\frac{2\pi}{2p+1}\right)\sin\left(\frac{4\pi}{2p+1}\right)\ldots\sin\left(\frac{2p\pi}{2p+1}\right).$$

Now, to make things easier to see, we will define a function $f$, where

$$f(n) = \sin\left(\frac{n\pi}{2p+1}\right).$$

This function has a special property: because $\sin(\pi - \theta) = \sin(\theta)$, we can say that:

$$f(n) = \sin\left(\frac{n\pi}{2p+1}\right)$$

$$= \sin\left(\pi - \frac{n\pi}{2p+1}\right)$$

$$= \sin\left(\frac{(2p+1-n)\pi}{2p+1}\right)$$

$$= f(2p+1-n).$$

We will use this on some of the terms in our product. Now, let $k$ be the integer such that $2k \leq p \leq 2k + 1$ (in other words, $k = \lfloor\frac{p}{2}\rfloor$). Then continuing from above, we have:

$$f(1)f(2)\ldots f(p)\times 2^p \times A$$

$$= f(2)f(4)\ldots f(2k)f(2(k+1))f(2(k+2))\ldots f(2p)$$

$$= f(2)f(4)\ldots f(2k)f(2p+1-2(k+1))f(2p+1-2(k+2))\ldots f(2p+1-2p)$$

$$= f(2)f(4)\ldots f(2k)f(2(p-k)-1)f(2(p-k-1)-1)\ldots f(2(1)-1).$$

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Now as $2k$ is the largest even integer less than or equal to $p$, $f(2)f(4)\ldots f(2k)$ covers all of the even numbers less than $p$. Also, we have $2k \leq p \leq 2k + 1$, which can be rearranged to give:

$$2k \leq p \Rightarrow p \leq 2(p - k)$$

and

$$p \leq 2k + 1 \Rightarrow 2(p - k) - 1 \leq p.$$

Thus $2(p - k) - 1 \leq p \leq 2(p - k)$, and hence $2(p - k) - 1$ is the largest odd integer less than or equal to $p$, so $f(2(p - k) - 1)f(2(p - k - 1) - 1)\ldots f(2(1) - 1)$ covers all the odd numbers less than $p$. Hence,

$$f(2)f(4)\ldots f(2k)f(2(p - k) - 1)f(2(p - k - 1) - 1)\ldots f(2(1) - 1) = f(1)f(2)\ldots f(p).$$

And, as this quantity can be shown not to be 0 (as $f(n) > 0$ for $0 < n < 2p + 1$), we have:

$$f(1)f(2)\ldots f(p) \times 2^p \times A = f(1)f(2)\ldots f(p) \Rightarrow 2^p A = 1 \Rightarrow 2^p \cos\left(\frac{\pi}{2p+1}\right) \cos\left(\frac{2\pi}{2p+1}\right) \ldots \cos\left(\frac{p\pi}{2p+1}\right) = 1,$$

as required!

7. First, we prove the following inequality:

$$\sqrt{k+1} \sqrt{a^2 + b^2} \geq a + b \sqrt{k},$$

where $a$, $b$ and $k$ are positive. This can be proved using the weighted QM-AM inequality, but here we do it using only the fact that $x^2 \geq 0$,

$$\frac{a\sqrt{k} - b}{2} \geq 0 \Rightarrow ka^2 - 2ab\sqrt{k} + b^2 \geq 0$$

$$ka^2 + b^2 \geq 2ab\sqrt{k}$$

$$(k + 1)\left(a^2 + b^2\right) \geq a^2 + 2ab\sqrt{k} + kb^2$$

$$\sqrt{k + 1} \sqrt{a^2 + b^2} \geq \left(a + b\sqrt{k}\right)^2$$

Notice that equality occurs when $b = a \sqrt{k}$.

For the sake of generality and convenience, we let $AB = d$.

Now we shall do the $n = 2$ case. We can walk from $A$ to $B$ in two straight line paths. Let the point where the two line segments meet be $C$. Let $AC$ extended be called line $\ell$. Let the perpendicular dropped from $B$ to $\ell$ meet $\ell$ at $P$. Note that $P$ is the closest point to $B$ on line $\ell$. $\angle BAC < 90^\circ$ since otherwise we would be walking away from $B$. Also, $\angle ACB \geq 90^\circ$, otherwise we would eventually pass $P$, and would thus be walking away from $B$ again. Thus, $C$ must lie somewhere on line segment $AP$. 

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By the triangle inequality, $CP + PB > CB$. Thus, to maximise the distance travelled we should walk all the way to $P$ first and then turn and walk to $B$ (in other words, we let $C = P$), meaning that $ACB$ is now a right-angled triangle. It only remains to determine the optimum shape of this triangle.

Let $AC = a$ and $CB = b$. By Pythagoras’ Theorem, $a^2 + b^2 = d^2$, and we wish to maximise $a + b$. By the inequality we proved at the beginning (with $k = 1$), we see that the maximum value is $\sqrt{k + 1}a^2 + b^2 = d\sqrt{2}$. Moreover, this is achievable by letting $a = b = d/\sqrt{2}$.

Now proceeding to the general case, we will show that the maximum distance is $d\sqrt{n}$, which is achieved by $n$ line segments each having a length of $d/\sqrt{n}$. We will do this by induction. Having already established this result for $n = 2$, we will assume it is true for $n = k$ and now use it to show it is also true for $n = k + 1$.

Take a path from $A$ to $B$ with $k + 1$ line segments. Let the point where we make our first turn be $C$. By the same argument as before, $\angle ACB \geq 90^\circ$. Define $a$ and $b$ as before. Note that this time we walk along $AC$, but then walk from $C$ to $B$ using the remaining $k$ line segments (i.e. not necessarily along the line segment $CB$). However, by our inductive hypothesis, we know that the maximum distance that this part of the walk can be is $b\sqrt{k}$. Thus, the total walk will be of length $a + b\sqrt{k}$.

Wherever $C$ is, if $\angle ACB < 90^\circ$ we can move $C$ vertically away from the line $AB$ until $\angle ACB = 90^\circ$. This increases both $a$ and $b$, and so will increase the total distance travelled.
Therefore, we again have that \( \triangle ACB \) is right-angled, and thus \( a^2 + b^2 = d^2 \). This time we wish to maximise \( a + b \sqrt{k} \), and we can do it by using the inequality from the beginning. The maximum value is \( d \sqrt{k+1} \). This is achieved by \( a \sqrt{k} = b \), and by using \( a^2 + b^2 = d^2 \) we obtain the length of the first leg of the journey, \( a = d / \sqrt{k+1} \). By our inductive hypothesis, the subsequent legs have length \( b / \sqrt{k} \), which is the same as \( a \), and thus all the legs have the same length. We have now established the result for \( n = k + 1 \), and so we have proved it for all natural numbers \( n \).

In our case, we have \( d = 10 \), so the maximum distance will be \( 10 \sqrt{n} \). The general path will consist of line segments having a length of \( 10 / \sqrt{n} \) and will have a spiral-like shape, with each point where a turn occurs forming a right-angle with the previous point and \( B \).

In particular, for \( n = 3 \) we have the maximum length being \( 10 \sqrt{3} \), and a path as shown below.