DEPARTMENT OR MATHEMATICS AND STATISTICS

620-161 INTRODUCTORY MATHEMATICS

MID-SEMESTER TEST

Semester 1, 2007
Time Allowed: 45 minutes

Student's Name (PRINT IN CAPITALS): ....

Student's No .... Tutor's Name: ....

Tutorial Time: .... Tutorial Place: ....

1. (a) (3 points) Write down the initial simplex tableau for the following linear programming problem.
DO NOT SOLVE THE PROBLEM!

$$\begin{aligned} &\text{Maximize P} = 2x_1 + 3x_2 - 4x_3 \\ &\text{subject to} \quad x_1 + x_2 + 2x_3 \le 25 \\ &\qquad\qquad\quad x_1 + 3x_2 + x_3 \le 43 \\ &\qquad\qquad\quad x_1, x_2, x_3 \ge 0 \end{aligned}$$

(b) (3 points) The following tableau arises when solving a linear programming problem. Obtain the next tableau, using the simplex method. Clearly circle your pivot element and state the row operations used.

BV	x_1	x_2	x_3	x_4	s_1	s_2	s_3	P	RHS
s_1	1	2	1	0	1	0	−1	0	3
s_2	1	−1	2	0	0	1	−1	0	9
x_4	0	1	0	1	0	0	1	0	6
P	−2	5	−4	0	0	0	6	1	11

(c) (2 points) Write down the solution represented by the following simplex tableau, which was used to maximize P.

BV	x_1	x_2	s_1	s_2	s_3	P	RHS
s_1	0	1	1	0	1	0	11
s_2	0	3	0	1	0	0	33
x_1	1	2	0	0	1	0	66
P	0	4	0	0	3	1	120

(d) (5 points) Write down the dual of the following linear program. DO NOT SOLVE.

$$\begin{array}{lrl} \text{Minimize} & P = & 3x_1 + 16x_2 \\ \text{subject to} & & x_1 - 3x_2 \geq 20 \\ & & -x_1 + x_2 \geq 30 \\ & & x_1, x_2 \geq 0 \end{array}$$

2. (a) (3 points) Solve for y in the equation $x + \log_e(2x) = \log(xy)$.

(b) (4 points) Find $f'(x)$ for each of the following cases:

(i) $f(x) = (x^3 + 4)\log_e 5x$ (ii) $f(x) = (e^x + 3x^2)^5$

(iii) $f(x) = \dfrac{3x - 4}{x^3 + 6}$ (iv) $f(x) = \cos^2(3x)$

3. (a) (4 points) Use Gauss-Jordan elimination (row operations) to find a solution of the following linear system. At each step, indicate carefully the row operations being undertaken.

$$\begin{aligned} x + 2y - z &= 1 \\ x + 3y + 3z &= -2 \\ 2x + 4y - 4z &= 2 \end{aligned}$$

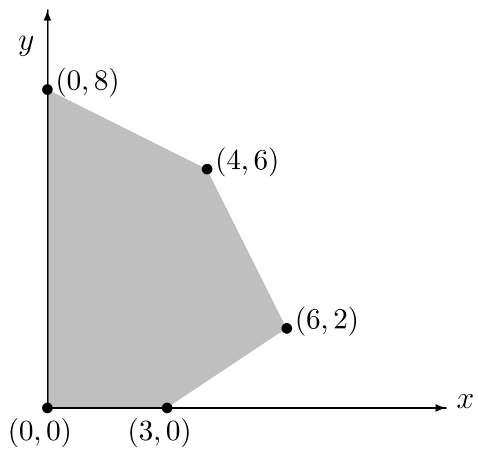
(b) (2 points) The following matrices arise when solving various systems of equations. State the complete solution the system has in each case. The variables corresponding to the columns from left to right are x, y and z.

$$(i) \quad \left[\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & 1 & -3 & 4 \\ 0 & 0 & 0 & 6 \end{array}\right] \qquad (ii) \quad \left[\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & 1 & -3 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

4. (a) (3 points) Sketch the region representing all feasible (x, y) values for the system below on the axes drawn below. Clearly label all corner points.

$$\begin{aligned} x + 2y &\le 8 \\ 3x + \ \ y &\le 9 \\ x, y &\ge 0 \end{aligned}$$

(b) (3 points) Given the feasible region shown shaded below, find the maximum value for $P = 4x + 2y$ and find the values of x and y where this maximum occurs.



(c) (1 point) List the extreme points of the feasible region given in (b).

(d) (2 points) How may optimal extreme points and how many optimal solutions are there in (b)?

Solution

1. (a) Write down the initial simplex tableau for the following linear programming problem.
DO NOT SOLVE THE PROBLEM!

Maximize P $= 2x_1 + 3x_2 - 4x_3$
subject to $x_1 + x_2 + 2x_3 \le 25$
$x_1 + 3x_2 + x_3 \le 43$
$x_1, x_2, x_3 \ge 0$

Solution:

BV	x_1	x_2	x_3	s_1	s_2	P	RHS
s_1	1	1	2	1	0	0	25
s_2	1	3	1	0	1	0	43
P	-2	-3	4	0	0	1	0

(b) The following tableau arises when solving linear programming problem. Obtain the next tableau, using the simplex method. Clearly circle your pivot element and state the row operations used.

BV	x_1	x_2	x_3	x_4	s_1	s_2	s_3	P	RHS
s_1	1	2	1	0	1	0	-1	0	3
s_2	1	-1	2	0	0	1	-1	0	9
x_4	0	1	0	1	0	0	1	0	6
P	-2	5	-4	0	0	0	6	1	11

Solution:

BV	x_1	x_2	x_3	x_4	s_1	s_2	s_3	P	RHS	RT		
s_1	1	2	**[1]**	0	1	0	-1	0	3	3	$\rightarrow$	
s_2	1	-1	2	0	0	1	-1	0	9	9/2		$-2R_1 + R_2 \rightarrow R_2$
x_4	0	1	0	1	0	0	1	0	6	$--$		
P	-2	5	-4	0	0	0	6	1	11			$4R_1 + R_4 \rightarrow R_4$
			$\uparrow$									

BV	x_1	x_2	x_3	x_4	s_1	s_2	s_3	P	RHS
x_3	1	2	1	0	1	0	-1	0	3
s_2	-1	-5	0	0	-2	1	1	0	3
x_4	0	1	0	1	0	0	1	0	6
P	2	13	0	0	4	0	2	1	23

(c) Write down the solution represented by the following simplex tableau, which was used to maximize P.

BV	x_1	x_2	s_1	s_2	s_3	P	RHS
s_1	0	1	1	0	1	0	11
s_2	0	3	0	1	0	0	33
x_1	1	2	0	0	1	0	66
P	0	4	0	0	3	1	120

Solution:
$(x_1, x_2, s_1, s_2, s_3, P) = (66, 0, 11, 33, 0, 120)$.

(d) Write down the dual of the following linear program. DO NOT SOLVE.

$$\begin{aligned} \text{Minimize} \quad P = {} & 3x_1 + 16x_2 \\ \text{subject to} \quad & x_1 - 3x_2 \geq 20 \\ & -x_1 + x_2 \geq 30 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Solution:

$$\begin{aligned} \text{Maximize} \quad W = {} & 20y_1 + 30y_2 \\ \text{subject to} \quad & y_1 - y_2 \leq 3 \\ & -3y_1 + y_2 \leq 16 \\ & y_1, y_2 \leq 0 \end{aligned}$$

2. (a) Solve for y in the equation $x + \log_e(2x) = \log(xy)$.
Solution:

$$\begin{aligned}
x + \log_e(2x) &= \log(xy) \\
x &= \log(xy) - \log_e(2x) \\
x &= \log_e\left(\frac{xy}{2x}\right) \\
x &= \log_e\left(\frac{y}{2}\right) \\
e^x &= e^{\log_e\left(\frac{y}{2}\right)} = \frac{y}{2} \\
y &= 2e^x
\end{aligned}$$

(b) Find $f'(x)$ for each of the following cases:

$f(x) = (x^3 + 4)\log_e 5x$

$f(x) = (e^x + 3x^2)^5$

$f(x) = \dfrac{3x - 4}{x^3 + 6}$

$f(x) = \cos^2(3x)$

Solution:

$$\left[\left(x^3+4\right)\log_e 5x\right]' = 3x^2\log_e(5x) + \left(x^3+4\right)\frac{1}{x}$$

$$\left[\left(e^x + 3x^2\right)^5\right]' = 5\left(e^x + 3x^2\right)^4\left[e^x + 6x\right]$$

$$\left[\frac{3x-4}{x^3+6}\right]' = \frac{3(x^3+6) - 3(3x-4)x^2}{\left[x^3+6\right]^2} = \frac{-6x^3 + 12x^2 + 18}{\left[x^3+6\right]^2}$$

$$\left[\cos^2(3x)\right]' = 2\cos(3x)\left(-\sin(3x)\right)(3) = -6\cos(3x)\sin(3x)$$

3. (a) Use Gauss-Jordan elimination (row operations) to find a solution of the following linear system. At each step, indicate carefully the row of operations being undertaken.

$$\begin{aligned}
x + 2y - z &= 1 \\
x + 3y + 3z &= -2 \\
2x + 4y - 4z &= 2
\end{aligned}$$

Solution:

$$\left[\begin{array}{rrr|r} 1 & 2 & -1 & 1 \\ 1 & 3 & 3 & -2 \\ 2 & 4 & -4 & 2 \end{array}\right] \begin{array}{l} \\ -R_1 + R_2 \to R_2 \\ -2R_1 + R_3 \to R_3 \end{array} \sim \left[\begin{array}{rrr|r} 1 & 2 & -1 & 1 \\ 0 & 1 & 4 & -3 \\ 0 & 0 & -2 & 0 \end{array}\right]$$

$$\left[\begin{array}{rrr|r} 1 & 2 & -1 & 1 \\ 0 & 1 & 4 & -3 \\ 0 & 0 & -2 & 0 \end{array}\right] \begin{array}{l} -2R_2 + R_1 \to R_1 \\ \\ \\ \end{array} \sim \left[\begin{array}{rrr|r} 1 & 0 & -9 & 7 \\ 0 & 1 & 4 & -3 \\ 0 & 0 & -2 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -9 & 7 \\ 0 & 1 & 4 & -3 \\ 0 & 0 & -2 & 0 \end{array}\right] \begin{array}{l} (-9/2)R_3 + R_1 \to R_1 \\ 2R_3 + R_2 \to R_2 \\ (-1/2)R_3 \to R_3 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Hence $(x, y, z) = (7, -3, 0)$

(b) The following matrices arise when solving various systems of equations. State the complete solution the system has in each case. The variables corresponding to the columns from left to right are x, y and z.

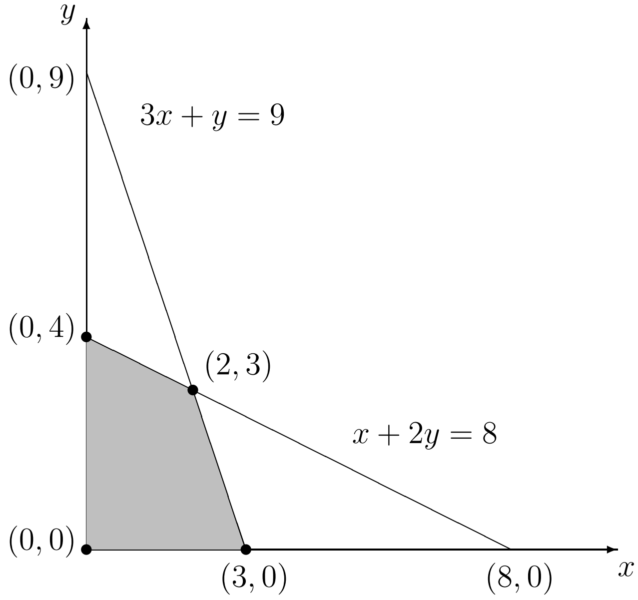
$$(i) \quad \left[\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & 1 & -3 & 4 \\ 0 & 0 & 0 & 6 \end{array}\right] \qquad (ii) \quad \left[\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & 1 & -3 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Solution:
(i) The system is inconsistent, there is no solution.
(ii) $(x, y, z) = (3 - t, 4 + 3t, t), \ t \in \mathbb{R}$.

4. (a) Sketch the region representing all feasible values (x, y) for the system below on the axes drawn below. Clearly label all corner points.

$$\begin{aligned} x + 2y &\leq 8 \\ 3x + \ y &\leq 9 \\ x, y &\geq 0 \end{aligned}$$

Solution:

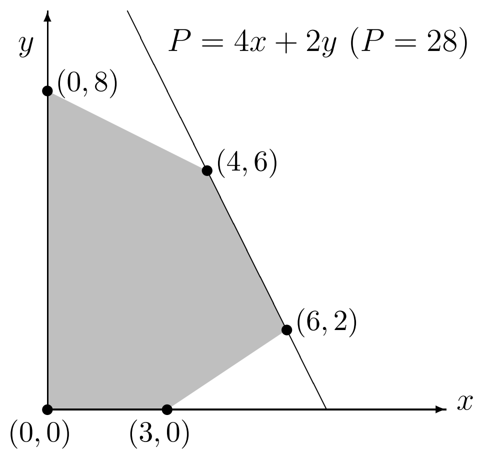


The corner point $(2, 3)$ is the intersection of the two lines representing the two constraints. Hence, it is the solution of the system

$$\begin{aligned} x + 2y &= 8 \\ 3x + \ y &= 9 \end{aligned}$$

(b) Given the feasible region shown shaded below, find the maximum value for $P = 4x + 2y$ and find the values of x and y where this maximum occurs.

Solution:
We simply evaluate the objective function, $P = 4x + 2y$, at all the corner points of the feasible region and select the one that yields the largest value of P.



$(x, y) = (0, 0) \rightarrow P = 0 + 0 = 0$
$(x, y) = (0, 8) \rightarrow P = 0 + 2(8) = 16$
$(x, y) = (4, 6) \rightarrow P = 4(4) + 2(6) = 28$
$(x, y) = (6, 2) \rightarrow P = 4(6) + 2(2) = 28$
$(x, y) = (3, 0) \rightarrow P = 4(3) + 0 = 12$

Thus, the largest value of P is 14 and the optimal values of (x, y) lie on the line segment connecting $(4, 6)$ and $(6, 2)$. There are infinitely many such points.

(c) (1 point) List the extreme points of the feasible region given in (b).

Solution:
The extreme points of the feasible region are $(0,0), (3,0), (6,2), (4,6), (0,8)$.

(d) (2 points) How may optimal extreme points and how many optimal solutions are there in (b)?

Solution:
There are 2 optimal extreme points and infinitely many optimal solutions.