

DEPARTMENT OF MATHEMATICS AND STATISTICS

620-161 INTRODUCTORY MATHEMATICS

MID-SEMESTER TEST

Semester 1, 2007

Time Allowed: 45 minutes

Student's Name (PRINT IN CAPITALS):

Student's No Tutor's Name:

Tutorial Time: Tutorial Place:

1. (a) (3 points) Sketch the region representing all feasible (x, y) values for the following system on the axes drawn below. Clearly label all corner points.

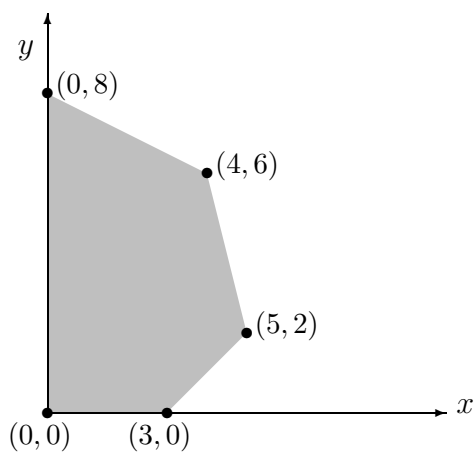
$$2x + 2y \leq 6$$

$$4x + 2y \leq 8$$

$$x, y \geq 0$$



- (b) (3 points) Given the feasible region shown shaded below, find the maximum value for $P = 8x + 2y$ and find the values of x and y where this maximum occurs.



- (c) (1 point) List the extreme points of the feasible region given in (b).
- (d) (2 points) How many optimal extreme points and how many optimal solutions are there in (b)?
2. (a) (4 points) Use Gauss-Jordan elimination (row operations) to find a solution of the following linear system. At each step, indicate carefully the row operations being undertaken.

$$\begin{aligned}x + 3y + z &= 11 \\-2x + 3y + 7z &= 5 \\6x + 10y + 8z &= 62\end{aligned}$$

- (b) (2 points) The following matrices arise when solving various systems of equations. State the complete solution the system has in each case. The variables corresponding to the columns from left to right are x , y and z .

$$(i) \left[\begin{array}{ccc|c} 1 & 1 & 3 & 4 \\ 0 & 7 & 1 & 2 \\ 0 & 0 & 0 & 9 \end{array} \right] \quad (ii) \left[\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

3. (a) (3 points) Solve for y in the equation $x + \log_e(2yx) = \log_e(x)$.

(b) (4 points) Find $f'(x)$ for each of the following cases:

$$f(x) = (x^4 + 3x) e^x$$

$$f(x) = \frac{3 \log_e x}{x^4}$$

$$f(x) = (2x^2 + 5)^8$$

$$f(x) = \sin^2(3x)$$

4. (a) (3 points) Write down the initial simplex tableau for the following linear programming problem.

DO NOT SOLVE THE PROBLEM!

$$\begin{aligned} \text{Maximize } P &= 4x_1 + 3x_2 - 7x_3 \\ \text{subject to } &2x_1 + x_2 + x_3 \leq 18 \\ &x_1 - x_2 + 2x_3 \leq 19 \\ &x_1, x_2 \geq 0 \end{aligned}$$

(b) (3 points) The following tableau arises when solving a linear programming problem. Obtain the next tableau, using the simplex method. Clearly circle your pivot element and state the row operations used.

<i>BV</i>	x_1	x_2	x_3	x_4	s_1	s_2	s_3	P	<i>RHS</i>
s_1	1	2	0	2	1	0	-1	0	6
s_2	1	-1	0	1	0	1	-1	0	2
x_3	0	1	1	0	0	0	1	0	4
P	-2	4	0	-4	0	0	5	1	11

- (c) (2 points) Write down the solution represented by the following simplex tableau, which was used to maximize P .

BV	x_1	x_2	s_1	s_2	s_3	P	RHS
x_2	0	1	0	1	1	0	4
x_1	1	0	0	0	1	0	7
s_1	0	0	1	1	0	0	3
P	0	0	0	6	5	1	22

- (d) (5 points) Write down the dual of the following linear program. DO NOT SOLVE.

$$\begin{aligned}
 &\text{Minimize } P = 4x_1 + 3x_2 \\
 &\text{subject to } \quad \quad \quad x_1 + x_2 \geq 1 \\
 &\quad \quad \quad \quad \quad -x_1 + 3x_2 \geq 4 \\
 &\quad \quad \quad \quad \quad x_1, x_2 \geq 0
 \end{aligned}$$

Solution

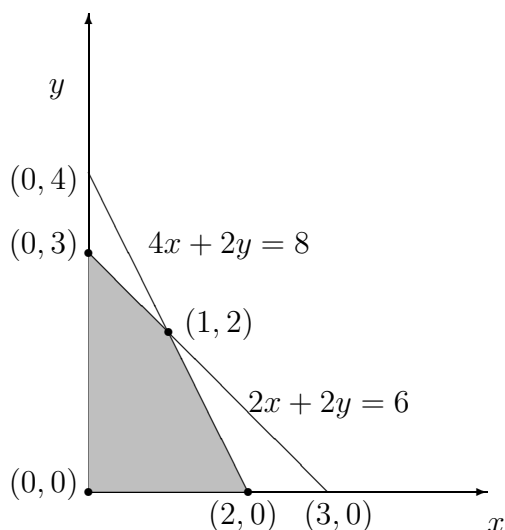
1. (a) Sketch the region representing all feasible values (x, y) for the system below on the axes drawn below. Clearly label all corner points.

$$2x + 2y \leq 6$$

$$4x + 2y \leq 8$$

$$x, y \geq 0$$

Solution:



The corner point $(1, 2)$ is the intersection of the two lines representing the two constraints. Hence, it is the solution of the system

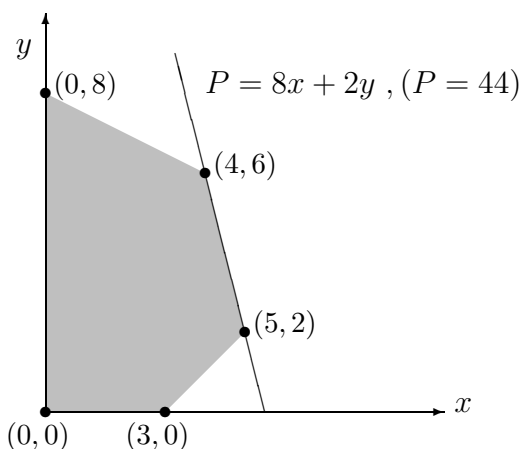
$$2x + 2y = 6$$

$$4x + 2y = 8$$

- (b) Given the feasible region shown shaded below, find the maximum value for $P = 8x + 2y$ and find the values of x and y where this maximum occurs.

Solution

We find the maximum value of P by evaluating the objective function at all the corner points of the feasible region. We then select the point(s) yielding the largest value of P .



$$(x, y) = (0, 0) \rightarrow P = 0 + 0 = 0$$

$$(x, y) = (0, 8) \rightarrow P = 0 + 2(8) = 16$$

$$(x, y) = (4, 6) \rightarrow P = 8(4) + 2(6) = 44$$

$$(x, y) = (5, 2) \rightarrow P = 8(5) + 2(2) = 44$$

$$(x, y) = (3, 0) \rightarrow P = 8(3) + 0 = 24$$

Hence, the largest value of P is 44 and is attained at any point on the line segment connecting $(4, 6)$ and $(5, 2)$. There are infinitely many optimal solutions.

- (c) (1 point) List the extreme points of the feasible region given in (b).

Solution:

The extreme points of the feasible region are $(0, 0)$, $(3, 0)$, $(5, 2)$, $(4, 6)$, $(0, 8)$.

- (d) (2 points) How many optimal extreme points and how many optimal solutions are there in (b)?

Solution:

There are 2 optimal extreme points and infinitely many optimal solutions.

2. (a) Use Gauss-Jordan elimination (row operations) to find a solution of the following linear system. At each step, indicate carefully the row operations being undertaken.

$$\begin{aligned}x + 3y + z &= 11 \\ -2x + 3y + 7z &= 5 \\ 6x + 10y + 8z &= 62\end{aligned}$$

Solution:

$$\begin{aligned}\left[\begin{array}{ccc|c} 1 & 3 & 1 & 11 \\ -2 & 3 & 7 & 5 \\ 6 & 10 & 8 & 62 \end{array} \right] & \begin{array}{l} 2R_1 + R_2 \rightarrow R_2 \\ -6R_1 + R_3 \rightarrow R_3 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 3 & 1 & 11 \\ 0 & 9 & 9 & 27 \\ 0 & -8 & 2 & -4 \end{array} \right] \\ \left[\begin{array}{ccc|c} 1 & 3 & 1 & 11 \\ 0 & 9 & 9 & 27 \\ 0 & -8 & 2 & -4 \end{array} \right] & \begin{array}{l} (-1/3)R_2 + R_1 \rightarrow R_1 \\ (1/9)R_2 \rightarrow R_2 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 0 & -2 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 10 & 20 \end{array} \right] \\ \left[\begin{array}{ccc|c} 1 & 0 & -2 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 10 & 20 \end{array} \right] & \begin{array}{l} (2/10)R_3 + R_1 \rightarrow R_1 \\ (-1/10)R_3 + R_2 \rightarrow R_2 \\ (1/10)R_3 \rightarrow R_3 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array} \right]\end{aligned}$$

Hence $(x, y, z) = (6, 1, 2)$.

- (b) The following matrices arise when solving various systems of equations. State the complete solution the system has in each case. The variables corresponding to the columns from left to right are x , y and z .

$$(i) \left[\begin{array}{ccc|c} 1 & 1 & 3 & 4 \\ 0 & 7 & 1 & 2 \\ 0 & 0 & 0 & 9 \end{array} \right] \quad (ii) \left[\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

Solution.

(i) The system is inconsistent, there are no solutions.

(ii) $(x, y, z) = (4 - t, 3 - 2t, t), t \in \mathbb{R}$.

3. (a) Solve for y in the equation $x + \log_e(2yx) = \log_e(x)$.

Solution:

$$\begin{aligned} x + \log_e(2yx) &= \log_e(x) \\ \log_e(2yx) - \log_e(x) &= -x \\ \log_e\left(\frac{2yx}{x}\right) &= -x \\ \log_e(2y) &= -x \\ e^{\log_e(2y)} &= e^{-x} \\ 2y &= e^{-x} \\ y &= \frac{e^{-x}}{2} \end{aligned}$$

- (b) Find $f'(x)$ for each of the following cases:

$$f(x) = (x^4 + 3x) e^x$$

$$f(x) = \frac{3 \log_e x}{x^4}$$

$$f(x) = (2x^2 + 5)^8$$

$$f(x) = \sin^2(3x) \text{ Solution:}$$

$$[(x^4 + 3x) e^x]' = (4x^3 + 3)e^x + (x^4 + 3x) e^x = e^x(x^4 + 4x^3 + 3x + 3)$$

$$\left[\frac{3 \log_e x}{x^4}\right]' = 3 \frac{\frac{1}{x} x^4 - 4x^3 \log_e x}{x^8} = \frac{3 - 12 \log_e x}{x^5}$$

$$[(2x^2 + 5)^8]' = 8(2x^2 + 5)^7 4x = 32x(2x^2 + 5)^7$$

$$[\sin^2(3x)]' = 2 \sin(3x) (\cos(3x)(3)) = 6 \sin(3x) \cos(3x)$$

4. (a) Write down an initial simplex tableau for the following linear programming problem.

DO NOT SOLVE THE PROBLEM!

$$\begin{aligned} \text{Maximize } P &= 4x_1 + 3x_2 - 7x_3 \\ \text{subject to } & 2x_1 + x_2 + x_3 \leq 18 \\ & x_1 - x_2 + 2x_3 \leq 19 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

Solution:

<i>BV</i>	x_1	x_2	x_3	s_1	s_2	P	<i>RHS</i>
s_1	2	1	1	1	0	0	18
s_2	1	-1	2	0	1	0	19
P	-4	-3	7	0	0	1	0

- (b) The following tableau arises when solving linear programming problem. Obtain the next tableau, using the simplex method. Clearly circle your pivot element and state the row operations used.

<i>BV</i>	x_1	x_2	x_3	x_4	s_1	s_2	s_3	P	<i>RHS</i>
s_1	1	2	0	2	1	0	-1	0	6
s_2	1	-1	0	1	0	1	-1	0	2
x_3	0	1	1	0	0	0	1	0	4
P	-2	4	0	-4	0	0	5	1	11

Solution:

<i>BV</i>	x_1	x_2	x_3	x_4	s_1	s_2	s_3	P	<i>RHS</i>	<i>RT</i>
s_1	1	2	0	2	1	0	-1	0	6	3 $-2R_2 + R_1 \rightarrow R_1$
s_2	1	-1	0	1	0	1	-1	0	2	2 \rightarrow
x_3	0	1	1	0	0	0	1	0	4	--
P	-2	4	0	-4	0	0	5	1	11	$4R_2 + R_4 \rightarrow R_4$

↑

<i>BV</i>	x_1	x_2	x_3	x_4	s_1	s_2	s_3	P	<i>RHS</i>
s_1	-1	4	0	0	1	-2	1	0	2
x_4	1	-1	0	1	0	1	-1	0	2
x_3	0	1	1	0	0	0	1	0	4
P	2	0	0	0	0	4	1	1	19

- (c) Write down the solution represented by the following simplex tableau, which was used to maximize P .

<i>BV</i>	x_1	x_2	s_1	s_2	s_3	P	<i>RHS</i>
x_2	0	1	0	1	1	0	4
x_1	1	0	0	0	1	0	7
s_1	0	0	1	1	0	0	3
P	0	0	0	6	5	1	22

Solution:

$$(x_1, x_2, s_1, s_2, s_3, P) = (4, 7, 3, 0, 0, 22).$$

- (d) Write down the dual of the following linear program. DO NOT SOLVE.

$$\begin{aligned} \text{Minimize } P &= 4x_1 + 3x_2 \\ \text{subject to } & x_1 + x_2 \geq 1 \\ & -x_1 + 3x_2 \geq 4 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Solution:

$$\begin{array}{ll} \text{Maximize } W = & y_1 + 4y_2 \\ \text{subject to} & y_1 - y_2 \leq 4 \\ & y_1 + 3y_2 \leq 3 \\ & y_1, y_2 \geq 0 \end{array}$$