620-161 SOLUTIONS TO HOMEWORK SHEET 4
Semester 1 2007

Out of 8 - Parts (a) and (b) each out of 4.

(a) *Simplex – dual.*

$$\begin{aligned} \text{Minimise} \quad & P = 4x_1 + 2x_2 + 5x_3 \\ \text{subject to} \quad & x_1 - x_2 + 2x_3 \geq 2 \\ & x_1 + 2x_2 + x_3 \geq 3 \\ & 2x_1 + x_2 - 2x_3 \geq 2 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

The dual problem to this is:

$$\begin{aligned} \text{Maximise} \quad & Q = 2y_1 + 3y_2 + 2y_3 \\ \text{subject to} \quad & y_1 + y_2 + 2y_3 \leq 4 \\ & -y_1 + 2y_2 + y_3 \leq 2 \\ & 2y_1 + y_2 - 2y_3 \leq 5 \\ & y_1, y_2, y_3 \geq 0 \end{aligned}$$

Rewrite this, introducing slack variables:

$$\begin{aligned} y_1 + y_2 + 2y_3 + s_1 \quad & = 4 \\ -y_1 + 2y_2 + y_3 \quad + s_2 \quad & = 2 \\ 2y_1 + y_2 - 2y_3 \quad + s_3 \quad & = 5 \\ -2y_1 - 3y_2 - 2y_3 \quad + Q & = 0 \qquad \text{(objective function)} \\ y_1, y_2, y_3, s_1, s_2, s_3 & \geq 0 \end{aligned}$$

Then rewrite in a simplex tableau:

BV	y_1	y_2	y_3	s_1	s_2	s_3	Q	RHS		ratio test:	
s_1	1	1	2	1	0	0	0	4		$\frac{4}{1} = 4$	$R_1 - R_2' \to R_1$
s_2	-1	$\boxed{2}$	1	0	1	0	0	2	$\longleftarrow$	$\frac{2}{2} = 1$	$\frac{1}{2}R_2 \to R_2'$
s_3	2	1	-2	0	0	1	0	5		$\frac{5}{1} = 5$	$R_3 - R_2' \to R_3$
Q	-2	-3	-2	0	0	0	1	0			$R_4 + 3R_2' \to R_4$
		$\uparrow$									

BV	y_1	y_2	y_3	s_1	s_2	s_3	Q	RHS		ratio test:	
s_1	$\frac{3}{2}$	0	$\frac{3}{2}$	1	$-\frac{1}{2}$	0	0	3		$\frac{3}{3/2} = 2$	$R_1 - \frac{3}{2}R_3' \to R_1$
y_2	$-\frac{1}{2}$	1	$\frac{1}{2}$	0	$\frac{1}{2}$	0	0	1		—	$R_2 + \frac{1}{2}R_3' \to R_2$
s_3	$\boxed{\frac{5}{2}}$	0	$-\frac{5}{2}$	0	$-\frac{1}{2}$	1	0	4	$\longleftarrow$	$\frac{4}{5/2} = \frac{8}{5}$	$\frac{2}{5}R_3 \to R_3'$
Q	$-\frac{7}{2}$	0	$-\frac{1}{2}$	0	$\frac{3}{2}$	0	1	3			$R_4 + \frac{7}{2}R_3' \to R_4$
	$\uparrow$										

BV	y_1	y_2	y_3	s_1	s_2	s_3	Q	RHS		ratio test:	
s_1	0	0	$\boxed{3}$	1	$-\frac{1}{5}$	$-\frac{3}{5}$	0	$\frac{3}{5}$	$\longleftarrow$	$\frac{3/5}{3} = \frac{9}{5}$	$\frac{1}{3}R_1 \to R_1'$
y_2	0	1	0	0	$\frac{2}{5}$	$\frac{1}{5}$	0	$\frac{9}{5}$		—	
y_1	1	0	-1	0	$-\frac{1}{5}$	$\frac{2}{5}$	0	$\frac{8}{5}$		—	$R_3 + R_1' \to R_3$
Q	0	0	-4	0	$\frac{4}{5}$	$\frac{7}{5}$	1	$8\frac{3}{5}$			$R_4 + 4R_1' \to R_4$
			$\uparrow$								

BV	y_1	y_2	y_3	s_1	s_2	s_3	Q	RHS
y_3	0	0	1	$\frac{1}{3}$	$-\frac{1}{15}$	$-\frac{1}{5}$	0	$\frac{1}{5}$
y_2	0	1	0	0	$\frac{2}{5}$	$\frac{1}{5}$	0	$\frac{9}{5}$
y_1	1	0	0	$\frac{1}{3}$	$-\frac{4}{15}$	$\frac{1}{5}$	0	$\frac{9}{5}$
Q	0	0	0	$\boxed{\frac{4}{3}}$	$\boxed{\frac{8}{15}}$	$\boxed{\frac{3}{5}}$	1	$9\frac{2}{5}$

Since there are no more negative entries in the bottom row we are done.

The optimal solution to the original minimisation problem is given by the slack variable entries in the last row as indicated, and the optimal values for the dual problems are equal, ie. $\min P = \max Q$. Therefore, the optimal solution to the original problem is

$$(x_1, x_2, x_3) = \Big(\frac{4}{3}, \frac{8}{15}, \frac{3}{5}\Big) \ ,$$

giving optimal value Min $P = 9\frac{2}{5}$.

CHECK:
We have

$$\text{Min } P = 4\Big(\frac{4}{3}\Big) + 2\Big(\frac{8}{15}\Big) + 5\Big(\frac{3}{5}\Big) = \frac{80+16+45}{15} = \frac{141}{15} = 9\frac{2}{5} \quad \checkmark$$

and substituting into the constraints:

$$\begin{aligned} \Big(\frac{4}{3}\Big) - \Big(\frac{8}{15}\Big) + 2\Big(\frac{3}{5}\Big) &= \frac{20-8+18}{15} = \frac{30}{15} = 2 \geq 2 \quad \checkmark \\ \Big(\frac{4}{3}\Big) + 2\Big(\frac{8}{15}\Big) + \Big(\frac{3}{5}\Big) &= \frac{20+16+9}{15} = \frac{45}{15} = 3 \geq 3 \quad \checkmark \\ 2\Big(\frac{4}{3}\Big) + \Big(\frac{8}{15}\Big) - 2\Big(\frac{3}{5}\Big) &= \frac{40+8-18}{15} = \frac{30}{15} = 2 \geq 2 \quad \checkmark \\ \text{and} \quad \frac{4}{3}, \frac{8}{15}, \frac{3}{5} &\geq 0 \quad \checkmark \end{aligned}$$

(b) *Log and Exp.*

$$
\begin{aligned}
&& \log_e(x+y) + x^2 + 3 &= \log_e(x^3+y) \\
\Rightarrow && x^2 + 3 &= \log_e(x^3+y) - \log_e(x+y) \\
&& &= \log_e\left(\frac{x^3+y}{x+y}\right) \\
\Rightarrow && e^{x^2+3} &= \frac{x^3+y}{x+y} \\
\Rightarrow && (x+y)e^{x^2+3} &= x^3+y \\
\Rightarrow && xe^{x^2+3} + ye^{x^2+3} &= x^3+y \\
\Rightarrow && ye^{x^2+3} - y &= x^3 - xe^{x^2+3} \\
\Rightarrow && y(e^{x^2+3} - 1) &= x^3 - xe^{x^2+3} \\
\Rightarrow && y &= \frac{x^3 - xe^{x^2+3}}{e^{x^2+3} - 1}
\end{aligned}
$$