# 620-161 SOLUTIONS TO HOMEWORK SHEET 6
Semester 1 2007

**Out of 8 - Parts (a) and (b) each out of 4.**

(a) (i) We have the equation:

$$x^2y - 2x^3 - y^3 + 1 = 0$$

Differentiating both sides with respect to $x$, we have:

$$\begin{aligned}
& \frac{d}{dx}\left(x^2y - 2x^3 - y^3 + 1\right) = \frac{d}{dx}(0) \\
\Rightarrow \quad & 2xy + x^2\frac{dy}{dx} - 6x^2 - 3y^2\frac{dy}{dx} = 0 \\
\Rightarrow \quad & \frac{dy}{dx}\left(x^2 - 3y^2\right) = 6x^2 - 2xy \\
\Rightarrow \quad & \frac{dy}{dx} = \frac{6x^2 - 2xy}{x^2 - 3y^2} \ .
\end{aligned}$$

At the given point $(2, -3)$, we have

$$\frac{dy}{dx}\Big|_{x=2,y=-3} = \frac{6x^2 - 2xy}{x^2 - 3y^2}\Big|_{x=2,y=-3} = \frac{24 + 12}{4 - 27} = -\frac{36}{23} \ ,$$

and so the slope of the curve at this point is $-\frac{36}{23}$.

(ii) We have the equation:

$$\frac{1}{x^2} + \frac{1}{y^2} = 5x$$

Differentiating both sides with respect to $x$, we have:

$$\begin{aligned}
& \frac{d}{dx}\left(\frac{1}{x^2} + \frac{1}{y^2}\right) = \frac{d}{dx}(5x) \\
\Rightarrow \quad & (-2)x^{-3} + (-2)y^{-3}\frac{dy}{dx} = 5 \\
\Rightarrow \quad & \frac{dy}{dx} = \frac{5 + \frac{2}{x^3}}{-\frac{2}{y^3}} \\
& = \frac{5x^3y^3 + 2y^3}{-2x^3} \ .
\end{aligned}$$

At the given point, we have

$$\frac{dy}{dx}\Big|_{x=1,y=\frac{1}{2}} = \frac{5x^3y^3 + 2y^3}{-2x^3}\Big|_{x=1,y=\frac{1}{2}} = \frac{\frac{5}{8} + \frac{2}{8}}{-2} = -\frac{7}{16} \ ,$$

that is, the slope of the curve at this point is $-\frac{7}{16}$.

(b) We have the function $f(x) = 3\cos(x) + (x+1)^3$. To find its Taylor polynomial of order 3 about $x = 0$ we need the following derivatives:

$$\begin{aligned}
f(x) &= 3\cos x + (x+1)^3 &\Rightarrow&\quad f(0) = 3\cos 0 + 1^3 = 3 + 1 = 4\\
f'(x) &= -3\sin x + 3(x+1)^2 &\Rightarrow&\quad f'(0) = -3\sin 0 + 3\cdot 1^2 = 0 + 3 = 3\\
f''(x) &= -3\cos x + 6(x+1) &\Rightarrow&\quad f''(0) = -3\cos 0 + 6\cdot 1 = -3 + 6 = 3\\
f'''(x) &= 3\sin x + 6 &\Rightarrow&\quad f'''(0) = 3\sin 0 + 6 = 0 + 6 = 6
\end{aligned}$$

Therefore the Taylor polynomial of order 3 about $x = 0$ is:

$$\begin{aligned}
f_3(x) &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3\\
&= 4 + 3x + \frac{3}{2}x^2 + \frac{6}{6}x^3\\
&= 4 + 3x + \frac{3}{2}x^2 + x^3\ .
\end{aligned}$$

An expression for the error at $x$ is

$$E_3(x) = \frac{f^{(4)}(c)}{4!}x^4 \qquad \text{for some } c \text{ between } 0 \text{ and } x\ ,$$

where

$$f^{(4)}(x) = 3\cos x\ .$$

So

$$E_3(x) = \frac{3\cos c}{4!}x^4\ .$$

When $x = 0.1$, we have:

$$\begin{aligned}
|E_3(0.1)| &= \left|\frac{3\cos c}{4!}(0.1)^4\right| &&\text{for some } c \text{ between } 0 \text{ and } 0.1\\
&< \left|\frac{3\cos 0}{4!}(0.1)^4\right| &&\text{since } \cos c \text{ is largest when } c = 0\\
&= \frac{3}{24}\cdot 0.0001\\
&= 0.0000125
\end{aligned}$$

as claimed.