620-161 SOLUTIONS TO HOMEWORK SHEET 7
Semester 1 2007

Out of 8 - Parts (a) and (b) each out of 4.

(a) (i) We have $f(x) = x^3 - x^2 - x + 2$. Stationary points of $f(x)$ occur where $f'(x) = 0$. So

$$f'(x) = 3x^2 - 2x - 1 = (3x+1)(x-1)\ .$$

So $f'(x) = 0$ when $x = -\frac{1}{3}$ or $x = 1$. Thus the only stationary point within the interval $(-1, 0)$ is

$$x = -\frac{1}{3}\ .$$

One way to determine the nature of this stationary point is to use the 2nd derivative test. We have $f''(x) = 6x - 2$, so

$$f''\big(-\frac{1}{3}\big) = 6\big(-\frac{1}{3}\big) - 2 = -4 < 0$$

so by the 2nd derivative test, $x = -\frac{1}{3}$ is a local maximum.

(ii) The absolute maximum and minimum values of $f(x)$ over $[-1, 0]$ may occur at either a stationary point of f within the interval, an endpoint of the interval, or a point in the interval where $f'(x)$ is undefined. Since $f'(x)$ is defined for all x, we compute

$$\begin{aligned} f\big(-\frac{1}{3}\big) &= \big(-\frac{1}{3}\big)^3 - \big(-\frac{1}{3}\big)^2 - \big(-\frac{1}{3}\big) + 2 = \frac{59}{27} = 2\frac{5}{27} \\ f(-1) &= (-1)^3 - (-1)^2 - (-1) + 2 = 1 \\ \text{and} \qquad f(0) &= 2\ . \end{aligned}$$

Therefore

$$\begin{aligned} &\text{the absolute maximum value of } f \text{ over } [-1,0] \text{ is } f\big(-\frac{1}{3}\big) = \frac{59}{27}\ , \\ \text{and}\quad &\text{the absolute minimum value of } f \text{ over } [-1,0] \text{ is } f(-1) = 1\ . \end{aligned}$$

(b) Let x (in centimeters) denote the side length of the square base of the box, and y (in centimeters) the height of the box. For the box to physically exist, $x, y > 0$. The volume of the box is

$$V = x^2 y = 500$$

which implies $y = 500/x^2$. The surface area of the box is

$$\begin{aligned} S &= x^2 + 4xy \\ &= x^2 + 4x\Big(\frac{500}{x^2}\Big) \\ &= x^2 + \frac{2000}{x} \end{aligned}$$

We want to minimise S on the interval $0 < x < \infty$.
To find stationary points of S we compute:

$$\frac{dS}{dx} = 2x - \frac{2000}{x^2} \ .$$

So the stationary points satisfy:

$$\begin{aligned} 2x - \frac{2000}{x^2} &= 0 \\ \Rightarrow \qquad 2x &= \frac{2000}{x^2} \\ \Rightarrow \qquad x^3 &= 1000 \\ \Rightarrow \qquad x &= 10 \ . \end{aligned}$$

We can use the second derivative test to check whether this is a minimum. We have

$$\frac{d^2S}{dx^2} = 2 + \frac{4000}{x^3}$$

which is positive for all $x > 0$. Thus $x = 10$ is a local minimum point. Moreover, since $x = 10$ is the only critical point of S in the interval $(0, \infty)$, it is also the global minimum.

When $x = 10$, $y = 500/(10)^2 = 5$, and so the dimensions of the box yielding minimum surface area are:

10cm by 10cm by 5cm

(the height being 5cm).

The minimum surface area is:

$$S(10) = 10^2 + \frac{2000}{10} = 300\,\text{cm}^2 \ .$$