

The University of Melbourne  
Department of Mathematics and Statistics  
620-161: Introductory Mathematics  
Problem Set #4 + Solution

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1. (a) Use Gauss-Jordan elimination to solve the following system of linear equations. At each step indicate clearly the row operations that you perform.

$$\begin{aligned} -x + 3y + 2z &= 8 \\ -2x + y - z &= 1 \\ 4x + 3y + 2z &= 23 \end{aligned}$$

- (b) Write down the solutions implied by the following augmented matrices. No row operations are required:

$$(i) \left[ \begin{array}{ccc|c} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 4 \end{array} \right] \quad (ii) \left[ \begin{array}{ccc|c} 1 & 4 & 8 & 2 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 0 & 2 \end{array} \right] \quad (iii) \left[ \begin{array}{ccc|c} 1 & 0 & 0 & -4 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

- (c) Briefly (no more than 3 lines) explain why it is clear that the following system is inconsistent (does not have a solution):

$$\begin{aligned} 3x + 2y + 3z &= 5 \\ 6x + 4y + 6z &= 6 \\ x + 3y + z &= 7 \end{aligned}$$

2. (a) Use the Graphical Method to solve the following linear programming problems, observing that feasible region is the same in both cases:

$$\begin{array}{ll} (i) & (ii) \\ \text{Maximize } P = 5x + 2y & \text{Minimize } P = 5x + 2y \\ \text{subject to } 3x + 2y \geq 12 & \text{subject to } 3x + 2y \geq 12 \\ 2x + 4y \leq 16 & 2x + 4y \leq 16 \\ y \leq 2 & y \leq 2 \\ x, y \geq 0 & x, y \geq 0 \end{array}$$

Observe that the first constraint is of the “ $\geq$ ” type. Shade the feasible region and clearly label all its corner points.

- (b) Write down a linear programming formulation for the problem below. Explain your notation clearly. **DO NOT ATTEMPT TO COMPUTE AN OPTIMAL SOLUTION.**

A contractor for the DIGUP Excavation Company mistakenly dug a hole in the wrong location. The contractor must now fill in the hole, at least to ground level, requiring at least 12 cubic metres of filling material. Because

the hole was dug next to a retaining wall, the filling must consist of sand and cement, with no more than four times as much sand as cement. Cement costs 10 times as much as sand per cubic metre.

How many cubic metres of sand and how many cubic metres of cement should the contractor use to minimize the cost of filling the hole?

3. (a) Use the Simplex Method to solve the following linear programming problem. At each step circle the pivot element and specify the row operations used.

$$\begin{aligned} \text{Maximize } P &= 8x_1 + 2x_2 + 6x_3 \\ \text{subject to } & 2x_1 + x_2 + x_3 \leq 1 \\ & 2x_1 + x_2 - x_3 \leq 6 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

- (b) (i) Write the dual of the following linear programming problem (**DO NOT COMPUTE THE OPTIMAL SOLUTION**).

$$\begin{aligned} \text{Minimize } C &= y_1 + 6y_2 \\ \text{subject to } & 2y_1 + 2y_2 \geq 8 \\ & y_1 + y_2 \geq 2 \\ & y_1 - y_2 \geq 6 \\ & y_1, y_2 \geq 0 \end{aligned}$$

- (ii) Is  $(y_1, y_2, C) = (6, 0, 6)$  an optimal solution to 3(b)? Justify your answer.

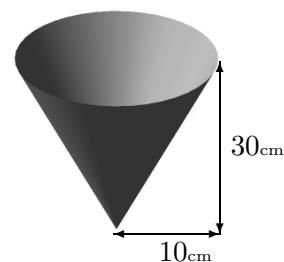
4. (a) Find the derivatives of the functions  $f$  and  $g$  that are defined as follows:

$$(i) \quad f(x) = \frac{\sin^2 x}{\log_e(x^2 + 1)} \qquad (ii) \quad g(x) = \left(x^2 + \sqrt{e^{2x} + 1}\right) \log_e(5x)$$

- (b) Juicy Problem:

Orange juice is dripping into a conical container from above at a constant rate of  $15 \text{ cm}^3$  per minute. The container has an inside height of 30 cm and an inside (top) radius of 10 cm.

Recall that the volume of a (right circular) cone of base radius  $r$  and height  $h$  is  $V = \frac{\pi hr^2}{3}$ .



- (i) Determine the rate of change in the depth of the juice in the container when the depth is equal to 11 cm.  
 (ii) If at time  $t = 0$  the container is empty, what will be the rate of change in the depth of the juice in the container after 60 minutes?

5. (a) Use implicit differentiation to find  $\frac{dy}{dx}$  at the point  $(x, y) = (1, 1)$  given that

$$e^y - 4x^2y^3 = y - 5 + e$$

- (b) (i) Construct the Taylor polynomial of order 2 about  $x = 0$  for the function  $f$  defined by

$$f(x) = e^{x^2+1}$$

- (ii) Write down an expression for the error  $E_2(x)$  when this polynomial is used to approximate  $f(x)$  at  $x = 0.1$ .

6. (a) Let  $f(x) = -x^3 + 3x^2 + 24x - 5$ .

- (i) Find all the critical points of  $f(x)$  in  $(-\infty, \infty)$ .  
(ii) Classify these points using the Second-Derivative Test.  
(iii) Confirm the classification using only first-derivatives (First Derivative Test).  
(iv) Find the global (absolute) maximum value and global (absolute) minimum value of  $f(x)$  in the interval  $[-1, 5]$ .

- (b) An artist makes a rectangular closed box with a square base. The box is to have a volume of 16,000 cubic centimeters. The material for the top and bottom of the box costs 3 cents per square centimeter, while the material for the sides costs 1.5 cents per square centimeter. Find the minimum total cost and the dimensions of the box that yield this cost.

7. (a) Let  $z = \log_e \sqrt{x^2 + y^2}$ .

(i) Find  $\frac{\partial z}{\partial x}$ ,  $\frac{\partial z}{\partial y}$ ,  $\frac{\partial^2 z}{\partial x^2}$ ,  $\frac{\partial^2 z}{\partial y^2}$  and  $\frac{\partial^2 z}{\partial x \partial y}$ .

(ii) Show that  $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$ .

- (iii) Find the equation of the tangent plane of the surface  $z = \log_e \sqrt{x^2 + y^2}$  at the point where  $(x, y) = (1, 1)$ .

- (b) A manufacturer of automobile batteries estimates that his total production in thousands of units is given by

$$f(x, y) = 3x^{\frac{1}{3}}y^{\frac{2}{3}} + 5,$$

where  $x$  is the number of units of labor and  $y$  is the number of units of capital utilized.

- (i) Find the total production when 64 units of labor and 125 units of capital are utilized.  
(ii) What would be the approximate effect on the total production if the labor is increased to 65 units and the capital is decreased to 124 units?

8. (a) Let

$$f(x, y) = x^2 + y^4 - 2y^2 - 20.$$

- (i) Find all the stationary points of  $f(x, y)$ .  
(ii) Classify these points using the Second-Derivative Test.  
(b) Use the Method of Lagrangian Multipliers to find a point  $(x, y)$  that minimizes  $f(x, y) = x^2 + 2y^2 - xy$  on the line  $x + y = 8$  and compute the minimum value of  $f(x, y)$ .

9. (a) Let  $z = 3 - 5i$  and  $w = -1 + 2i$ . Express the following complex numbers in the form  $a + bi$ , where  $a$  and  $b$  are real numbers.

(i)  $z + w$ ;

(ii)  $4z - 3w$ ;

(iii)  $z^2w$ ;

(iv)  $\frac{w}{z + w}$ .

(b) Solve the following equations:

(i)  $2z^2 + 6iz - 3 = 0$ ;

(ii)  $z^3 - 3z^2 + 5z - 3 = 0$ .

10. (a) Consider the following differential equation:

$$\frac{dy}{dx} = 3x^2y - 2xy.$$

(i) Show that  $y = Ae^{x^3-x^2}$  is the general solution of this equation, where  $A$  is an arbitrary constant.

(ii) Find the particular solution of this equation which satisfies the initial condition  $y(2) = \frac{1}{2}$ .

(b) Find the general solutions of the following differential equations:

(i)  $\frac{dy}{dx} = 3e^{-2x} + \frac{x}{x^2 + 1} + \sin x$ .

(ii)  $\frac{dy}{dx} = \frac{e^{3x} - x^2 + 5}{y^2 + 1}$ .

## Solution

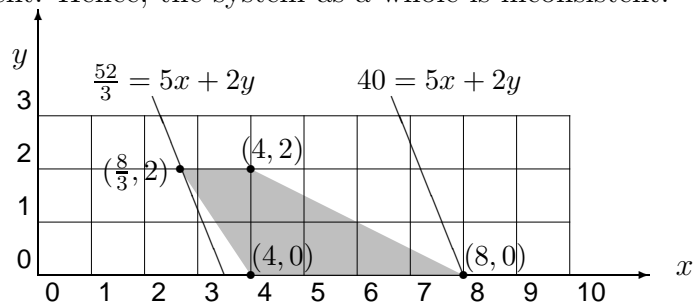
1. (a)

$$\begin{aligned} \left[ \begin{array}{ccc|c} -1 & 3 & 2 & 8 \\ -2 & 1 & -1 & 1 \\ 4 & 3 & 2 & 23 \end{array} \right] & \begin{array}{l} -R_1 \rightarrow R_1 \\ -2R_1 + R_2 \rightarrow R_2 \\ 4R_1 + R_3 \rightarrow R_3 \end{array} \sim \left[ \begin{array}{ccc|c} 1 & -3 & -2 & -8 \\ 0 & -5 & -5 & -15 \\ 0 & 15 & 10 & 55 \end{array} \right] \\ \left[ \begin{array}{ccc|c} 1 & -3 & -2 & -8 \\ 0 & -5 & -5 & -15 \\ 0 & 15 & 10 & 55 \end{array} \right] & \begin{array}{l} (-3/5)R_2 + R_2 \rightarrow R_2 \\ (-1/5)R_2 \rightarrow R_2 \\ 3R_2 + R_3 \rightarrow R_3 \end{array} \sim \left[ \begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & -5 & 10 \end{array} \right] \\ \left[ \begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & -5 & 10 \end{array} \right] & \begin{array}{l} (-1/5)R_3 + R_2 \rightarrow R_2 \\ (1/5)R_3 + R_2 \rightarrow R_2 \\ (-1/5)R_3 \rightarrow R_3 \end{array} \sim \left[ \begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & -2 \end{array} \right] \end{aligned}$$

So  $(x, y, z) = (3, 5, -2)$ . Students should check the results ...

(b) (i)  $(x, y, z) = (2, 1, 4)$  (ii) No solution. (iii)  $(x, y, z) = (-4, 1-t, t)$ ,  $t \in \mathbb{R}$ .

(c) The LHS of the second equation is twice the LHS of the first equation but the respective RHSs are not related this way so these two constraints are inconsistent. Hence, the system as a whole is inconsistent.



2. (a) (i) The optimal solution is the intersection of  $y = 0$  and  $2x + 3y = 16$ , yielding  $(x, y, P) = (8, 0, 40)$ .  
 (ii) The optimal solution is the intersection of  $y = 2$  and  $3x + 2y = 12$ , yielding  $(x, y, P) = (8/3, 4, 52/3)$ .

(b) Let  $x$  denote the volume (cubic metres) of sand used, and let  $y$  denote the volume (cubic metres) of concrete used to fill in the hole.

Also, let  $c$  denote the cost of one cubic metre of sand.

Objective function (cost of material):  $cx + 10cy$

Constraints:

$$\text{Total volume: } x + y \geq 12$$

$$\text{Sand/Cement ratio: } x/y \leq 4 \implies x - 4y \leq 0$$

Hence, the problem can be formulated as follows:

$$\begin{aligned} \text{Minimize } & cx + 10cy \\ \text{subject to } & x + y \geq 12 \\ & x - 4y \leq 0 \\ & x, y \geq 0 \end{aligned}$$

3. (a) Adding slack variables, the problem can be rewritten as follows:

$$\begin{aligned} \text{Maximize } P &= 8x_1 + 2x_2 + 6x_3 \\ \text{subject to } 2x_1 + x_2 + x_3 + s_1 &= 1 \\ 2x_1 + x_2 - x_3 + s_2 &= 6 \\ x_1, x_2, x_3, s_1, s_2 &\geq 0 \end{aligned}$$

So here is the Initial Simplex Tableau, followed by subsequent tableaus:

BV	$x_1$	$x_2$	$x_3$	$s_1$	$s_2$	RHS	RT
$s_1$	2	1	1	1	0	1	$1/2 * (1/2)R_1 \rightarrow R_1$
$s_2$	2	1	-1	0	1	6	$R_2 - R_1 \rightarrow R_2$
$P$	-8*	-2	-6	0	0	0	$R_3 + 4R_1 \rightarrow R_3$

BV	$x_1$	$x_2$	$x_3$	$s_1$	$s_2$	RHS	RT
$x_1$	1	$1/2$	$1/2$	$1/2$	0	$1/2$	$1 * 2R_1 \rightarrow R_1$
$s_2$	0	0	-2	-1	1	5	$4R_1 + R_2 \rightarrow R_2$
$P$	0	2	-2*	4	0	4	$4R_1 + R_2 \rightarrow R_2$

BV	$x_1$	$x_2$	$x_3$	$s_1$	$s_2$	RHS
$x_3$	2	1	1	1	0	1
$s_2$	4	2	0	1	1	7
$P$	4	4	0	6	0	6

So  $x^* = (0, 0, 1)$ ,  $s^* = (0, 7)$ ,  $P^* = 6$ . Students should check the results ...

- (b) (i) The same as the problem in 3(a).  
(ii) From duality theory of LP, since  $y = (6, 0)$  is feasible for the Minimization problem and the  $C = y_1 + 6y_2 = 6 + 1(0) = 6$  is equal to the optimal  $P$  in the Dual problem in 3(a), it follows that  $(6, 0)$  is an optimal solution for the Minimization problem.

Indeed, from the final Simplex Tableau of the Dual problem (reduced costs of the slack variables), we see that  $y = (6, 0)$  is an optimal solution to the Minimization problem.

4. (a)

$$\begin{aligned} (i) \quad f'(x) &= \left[ \frac{\sin^2 x}{\log_e(x^2 + 1)} \right]' \\ &= \frac{[\sin^2 x]' [\log_e(x^2 + 1)] - [\sin^2 x] [\log_e(x^2 + 1)]'}{[\log_e(x^2 + 1)]^2} \\ &= \frac{[2 \sin(x) \cos x] [\log_e(x^2 + 1)] - [\sin^2 x] \left[ \frac{2x}{x^2 + 1} \right]}{[\log_e(x^2 + 1)]^2} \\ &= \frac{[2 \sin(x) \cos x] [\log_e(x^2 + 1)] (x^2 + 1) - 2x \sin^2 x}{(x^2 + 1) [\log_e(x^2 + 1)]^2} \end{aligned}$$

$$\begin{aligned}
(ii) \quad f'(x) &= \left[ (x^2 + \sqrt{e^{2x} + 1}) \log_e(5x) \right]' \\
&= \left[ (x^2 + \sqrt{e^{2x} + 1}) \right]' [\log_e(5x)] + \left[ (x^2 + \sqrt{e^{2x} + 1}) \right] [\log_e(5x)]' \\
&= \left[ \left( 2x + \frac{e^{2x}}{\sqrt{e^{2x} + 1}} \right) \right] [\log_e(5x)] + \left[ (x^2 + \sqrt{e^{2x} + 1}) \right] \left[ \frac{1}{x} \right] \\
&= \left[ \left( 2x + \frac{e^{2x}}{\sqrt{e^{2x} + 1}} \right) \right] [\log_e(5x)] + \frac{x^2 + \sqrt{e^{2x} + 1}}{x}
\end{aligned}$$

(b) Juicy Problem:

(i) From the specification of the problem:

$$\begin{aligned}
V &= \frac{\pi h r^2}{3} \\
\frac{r}{h} &= \frac{10}{30} = \frac{1}{3} \implies r = \frac{h}{3}
\end{aligned}$$

Hence,

$$\begin{aligned}
V &= \frac{\pi h \left(\frac{h}{3}\right)^2}{3} = \frac{\pi h^3}{27} \\
\frac{d}{dt}V &= \frac{d}{dt} \left[ \frac{\pi h^3}{27} \right] = \frac{\pi}{27} 3h^2 \frac{dh}{dt} = \frac{\pi h^2}{9} \frac{dh}{dt}
\end{aligned}$$

Therefore

$$\begin{aligned}
\frac{dh}{dt} &= \frac{9 \frac{d}{dt}V}{\pi h^2} = \frac{9(15)}{\pi h^2} \\
&= \frac{135}{\pi h^2} \tag{***}
\end{aligned}$$

Hence,

$$\left. \frac{dh}{dt} \right|_{h=11} = \left. \frac{135}{\pi h^2} \right|_{h=11} = \frac{135}{121\pi} \text{ (cm}^3/\text{min)}$$

(ii) Since the filling rate is  $15 \text{ cm}^2$  per minute, after 60 minutes the juice will occupy a volume of  $V = 60 \cdot 15 = 900 \text{ cm}^2$ , corresponding to a height  $h$  such that

$$900 = \frac{\pi h r^2}{3} \implies h = \frac{2700}{\pi r^2} = \frac{2700}{100\pi} = \frac{27}{\pi}$$

Substituting this value of  $h$  in (\*\*\*) we obtain

$$\left. \frac{dh}{dt} \right|_{t=60} = \frac{135}{\pi h^2} = \frac{135}{\pi \left(\frac{27}{\pi}\right)^2} = \frac{135\pi}{27^2} = \frac{5\pi}{27}$$

5. (a)

$$\begin{aligned}
\frac{d}{dx} [e^y - 4x^2y^3] &= \frac{d}{dx} [y - 5 + e] \\
\frac{d}{dx} e^y - \frac{d}{dx} [4x^2y^3] &= \frac{dy}{dx} \\
e^y \frac{dy}{dx} - \frac{d}{dx} [4x^2y^3] &= \frac{dy}{dx} \\
e^y \frac{dy}{dx} - \left[ 8xy^3 + 12x^2y^2 \frac{dy}{dx} \right] &= \frac{dy}{dx} \\
\frac{dy}{dx} [e^y - 12x^2y^2 - 1] &= 8xy^3 \\
\frac{dy}{dx} &= \frac{8xy^3}{e^y - 12x^2y^2 - 1} \\
\left. \frac{dy}{dx} \right|_{(x,y)=(1,1)} &= \left. \frac{8xy^3}{e^y - 12x^2y^2 - 1} \right|_{(x,y)=(1,1)} \\
&= \frac{8(1)(1)^3}{e^1 - 12(1)^2(1)^2 - 1} \\
&= \frac{8}{e - 13}
\end{aligned}$$

(b) (i)

$$\begin{aligned}
f(x) &= e^{x^2+1} && \implies f(0) = e \\
f'(x) &= 2xe^{x^2+1} && \implies f'(0) = 0 \\
f''(x) &= 2e^{x^2+1} + 4x^2e^{x^2+1} && \implies f''(0) = 2e \\
f'''(x) &= 4xe^{x^2+1} + 8xe^{x^2+1} + 8x^3e^{x^2+1} \\
&= 4xe^{x^2+1}(3 + 2x^2) \\
p_2(x) &= f(0) + f'(0)x + \frac{f''(0)x^2}{2} = e + 2ex^2
\end{aligned}$$

(ii)

$$E_2(x) = \frac{f'''(c)x^3}{6} = \frac{x^3 [4ce^{c^2+1}(3 + 2c^2)]}{6}$$

for some  $0 \leq c \leq x, x \geq 0$ . Thus,

$$E_2(0.1) = \frac{(0.1)^3 [4ce^{c^2+1}(3 + 2c^2)]}{6}$$

for some  $0 \leq c \leq 0.1$ .

Because the expression is increasing with  $c \geq 0$ , it follows that the critical value of  $c$  is  $c = 0.1$ . Hence,

$$\begin{aligned}
E_2(0.1) &\leq \frac{(0.1)^3 [4(0.1)e^{(0.1)^2+1}(3 + 2(0.01))]}{6} \\
&= \frac{0.001 [0.4e^{1.01}(3.02)]}{6} = \frac{0.000604e^{1.01}}{3}
\end{aligned}$$

6. (a) (i) We have  $f'(x) = -3x^2 + 6x + 24 = -3(x^2 - 2x - 8) = -3(x+2)(x-4)$ . Setting  $f'(x) = 0$  we get two stationary points, namely  $x = -2$  and  $x = 4$ .
- (ii) Using the Second-derivative Test:  $f''(x) = -6x + 6$ ,  $f''(-2) = 18 > 0$ ,  $f''(4) = -18 < 0$ . By the second derivative test,  $x = -2$  is a local minimum point and  $x = 4$  is a local maximum point.
- (iii) The sign of  $f'(x)$  is given by the following diagram:

$x$	$(-\infty, -2)$	$-2$	$(-2, 4)$	$4$	$(4, \infty)$
$f'(x)$	-	0	+	0	-
$f(x)$	↘		↗		↘

By the First-derivative Test it follows that  $x = -2$  is a local minimum point and  $x = 4$  is a local maximum point over  $(-\infty, \infty)$ .

- (iv) The point  $x = 4$  is the only stationary point in the interval  $[-1, 5]$ . Evaluating  $f(x)$  at  $4, -1, 5$  we have  $f(4) = 75, f(-1) = -25, f(5) = 65$ . Hence the global maximum value of  $f$  on  $[-1, 5]$  is  $f(4) = 75$  and the global minimum value is  $f(-1) = -25$ .
- (b) Let  $x$  be the dimension of the base, and  $y$  the height of the box. Then  $x, y > 0$ ,  $x^2y = 16,000$ , and hence  $y = 16,000/x^2$ . Let  $C$  be the total cost (in cents). Then  $C = 2 \times 3 \times x^2 + 4 \times 1.5 \times xy = 6x^2 + 6xy = 6(x^2 + \frac{16,000}{x})$ , which is a function of  $x$ . We have

$$\frac{dC}{dx} = 6(2x - \frac{16,000}{x^2}).$$

Setting  $dC/dx = 0$ , we have  $x = 8,000/x^2$ . Solving this equation we get  $x = 20$ , and hence  $x = 20$  is the unique stationary point of  $C(x)$  in  $(0, \infty)$ . Note that  $dC/dx < 0$  when  $0 < x < 20$  and hence  $C(x)$  decreases in  $(0, 20)$ . Similarly,  $dC/dx > 0$  when  $20 < x < \infty$ , and hence  $C(x)$  increases in  $(20, \infty)$ . Therefore,  $x = 20$  is the unique global minimum point in  $(0, \infty)$ . When  $x = 20$  we have  $y = 16,000/x^2 = 40$ . Therefore, the minimum total cost is achieved when the base has dimension  $x = 20$  and the height is  $y = 40$ . The minimum total cost is  $6 \times 20^2 + 6 \times 20 \times 40 = 72,00$  cents.

7. (a) (i) We have

$$\begin{aligned} \frac{\partial z}{\partial x} &= \frac{x}{x^2 + y^2}, & \frac{\partial z}{\partial y} &= \frac{y}{x^2 + y^2} \\ \frac{\partial^2 z}{\partial x^2} &= \frac{y^2 - x^2}{(x^2 + y^2)^2}, & \frac{\partial^2 z}{\partial y^2} &= \frac{x^2 - y^2}{(x^2 + y^2)^2} \\ \frac{\partial^2 z}{\partial x \partial y} &= -\frac{2xy}{(x^2 + y^2)^2}. \end{aligned}$$

- (ii) From the expressions of  $\partial^2 z / \partial x^2$  and  $\partial^2 z / \partial y^2$  it is clear that

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0.$$

(iii) From (i) we have

$$\frac{\partial z}{\partial x}\Big|_{(x,y)=(1,1)} = \frac{\partial z}{\partial y}\Big|_{(x,y)=(1,1)} = \frac{1}{2}.$$

Also, when  $(x, y) = (1, 1)$  we have  $z = \log_e \sqrt{2} = (\log_e 2)/2$ . Thus, the equation of the tangent plane at the point when  $(x, y) = (1, 1)$  is

$$z - \frac{\log_e 2}{2} = \frac{1}{2}(x - 1) + \frac{1}{2}(y - 1)$$

i.e.,

$$x + y - 2z = 2 - \log_e 2.$$

- (b) Since  $f(x, y) = 3x^{\frac{1}{3}}y^{\frac{2}{3}} + 5$ , we have  $f(64, 125) = 3 \cdot 64^{\frac{1}{3}} \cdot 125^{\frac{2}{3}} + 5 = 305$ . That is, the total production is 305 thousands of units when  $(x, y) = (64, 125)$ . We have  $\frac{\partial f}{\partial x} = x^{-\frac{2}{3}}y^{\frac{2}{3}}$ ,  $\frac{\partial f}{\partial y} = 2x^{\frac{1}{3}}y^{-\frac{1}{3}}$ . Hence

$$\frac{\partial f}{\partial x}\Big|_{x=64, y=125} = \frac{25}{16}, \quad \frac{\partial f}{\partial y}\Big|_{x=64, y=125} = \frac{8}{5}.$$

The changes of  $x, y$  are

$$\Delta x = 1, \quad \Delta y = -1.$$

Thus, the approximate change of total production is given by

$$\begin{aligned} \Delta f &\approx \frac{\partial f}{\partial x}\Big|_{x=64, y=125} \Delta x + \frac{\partial f}{\partial y}\Big|_{x=64, y=125} \Delta y \\ &= \frac{25}{16} \times 1 + \frac{8}{5} \times (-1) \\ &= \frac{61}{80} \text{ (thousands of units) .} \end{aligned}$$

That is, the total production would be increased by 61/80 thousands of units approximately.

8. (a) We have  $f_x = 2x$ ,  $f_y = 4y^3 - 4y$ . Setting  $f_x = f_y = 0$  we get

$$2x = 0, \quad y^3 - y = 0.$$

Hence  $x = 0$ , and  $y = 0, -1$  or  $1$ . Thus, there are three stationary points:  $(0, 0)$ ,  $(0, -1)$  and  $(0, 1)$ .

We have

$$A = f_{xx} = 2, \quad B = f_{xy} = 0, \quad C = f_{yy} = 12y^2 - 4.$$

At  $(0, 0)$  we have  $A = 2, B = 0, C = -4$  and thus  $AC - B^2 = -8 < 0$ . Hence  $(0, 0)$  is a saddle point.

At  $(0, -1)$  we have  $A = 2 > 0, B = 0, C = 8$  and so  $AC - B^2 = 16 = 108 > 0$ . Hence  $(0, -1)$  is a local minimum point.

Similarly, at  $(0, 1)$  we have  $A = 2 > 0, B = 0, C = 8$  and  $AC - B^2 = 16 = 108 > 0$ . Hence  $(0, 1)$  is a local minimum point.

- (b) Let  $f(x, y) = x^2 + 2y^2 - xy$  and  $g(x, y) = x + y - 8$ . The problem is to maximize  $f(x, y)$  subject to  $g(x, y) = 0$ . Let

$$F(x, y, \lambda) = f(x, y) + \lambda g(x, y) = (x^2 + 2y^2 - xy) + \lambda(x + y - 8).$$

Then

$$F_x = 2x - y + \lambda, \quad F_y = -x + 4y + \lambda, \quad F_\lambda = x + y - 8.$$

Set

$$\begin{aligned} 2x - y + \lambda &= 0 \\ -x + 4y + \lambda &= 0 \\ x + y - 8 &= 0. \end{aligned}$$

Solving this equation system (Gaussian elimination, substitution, partial substitution, etc.), we get a unique solution:

$$x = 5, y = 3, \lambda = -7.$$

Since this is the unique stationary point,  $(x, y) = (5, 3)$  must be the unique minimum point. The minimum value of  $f(x, y)$  under the constraint is  $f(5, 3) = 5^2 + 2 \times 3^2 - 5 \times 3 = 28$ .

9. (a) (i)  $z + w = (3 - 1) + (-5 + 2)i = 2 - 3i$ .  
(ii)  $4z - 3w = (12 - 20i) - (-3 + 6i) = (12 + 3) - (20 + 6)i = 15 - 26i$ .  
(iii)  $z^2w = (3 - 5i)(3 - 5i)(-1 + 2i) = (9 - 30i - 25)(-1 + 2i) = (-16 - 30i)(-1 + 2i) = 16 - 32i + 30i + 60 = 76 - 2i$   
(iv)  $\frac{w}{z+w} = \frac{-1+2i}{2-3i} = \frac{(-1+2i)(2+3i)}{(2-3i)(2+3i)} = \frac{-8+i}{2^2+3^2} = -\frac{8}{13} + \frac{1}{13}i$ .
- (b) (i) By completing the square we get  $(z + \frac{3i}{2})^2 = -\frac{3}{4}$ , which gives  $z + \frac{3i}{2} = \pm \frac{\sqrt{3}}{2}i$ . Thus,  $z_1 = \frac{-3+\sqrt{3}}{2}i, z_2 = \frac{-3-\sqrt{3}}{2}i$ .  
Another method: By the quadratic formula we get

$$z = \frac{-6i \pm \sqrt{(6i)^2 - 4 \times 2 \times (-3)}}{2 \times 2} = \frac{-3 \pm \sqrt{3}}{2}i.$$

- (ii) Denote  $f(z) = z^3 - 3z^2 + 5z - 3$ . Observe that  $f(1) = 0$  and so  $z - 1$  is a factor of  $f(z)$ . Using long division we get  $f(z) = (z - 1)(z^2 - 2z + 3)$ . So the given equation can be written as  $(z - 1)(z^2 - 2z + 3) = 0$ . Thus,  $z = 1$  or  $z^2 - 2z + 3 = 0$ . For the latter one, by completing the square we have  $(z - 1)^2 = -2$ , which gives  $z - 1 = \pm\sqrt{2}i$ , that is,  $z = 1 \pm \sqrt{2}i$ . Thus, the solutions are  $z_1 = 1, z_2 = 1 + \sqrt{2}i, z_3 = 1 - \sqrt{2}i$ .

Quadratic formula for  $z^2 - 2z + 3 = 0$ :  $z = \frac{2 \pm \sqrt{(-2)^2 - 4 \times 1 \times 3}}{2} = \frac{2 \pm 2\sqrt{2}i}{2} = 1 \pm \sqrt{2}i$ .

10. (a) (i) For  $y = Ae^{x^3-x^2}$  with  $A$  an arbitrary constant, we have

$$\frac{dy}{dx} = Ae^{x^3-x^2}(3x^2 - 2x) = y(3x^2 - 2x) = 3x^2y - 2xy.$$

So  $y = Ae^{x^3-x^2}$  is the general solution to the given differential equation.

- (ii) If  $y(2) = 1/2$ , then  $1/2 = Ae^{2^3-2^2} = Ae^4$  and hence  $A = e^{-4}/2$ . The particular solution satisfying  $y(2) = 1/2$  is  $y = e^{x^3-x^2-4}/2$ .
- (b) (i)  $y = -\frac{3}{2}e^{-2x} + \frac{1}{2}\log_e(x^2 + 1) - \cos x + C$ , where  $C$  is an arbitrary constant.
- (ii) The given equation is separable:

$$(y^2 + 1)dy = (e^{3x} - x^2 + 5)dx.$$

Taking antiderivative w.r.t  $x$  on both sides we get:

$$\frac{1}{3}y^3 + y = \frac{1}{3}e^{3x} - \frac{1}{3}x^3 + 5x + \frac{1}{3}C,$$

where  $C$  is an arbitrary constant. Hence the (implicit) general solution is

$$y^3 + 3y = e^{3x} - x^3 + 15x + C.$$