**Cutting Plane Method for IP Problems**  
(*Winston Section 9.8*)

- **Observation:**
  If the LP relaxation of the IP problem has an integer optimal solution, then this solution is also optimal for the IP problem. Thus, it makes sense to force (legally!) the RHS values in the simplex tableau to be integers.

**Motivation**

- If all the variables are required to be integers and all the coefficients of the constraints are integers, then clearly these constraints will take integer values.
- In such cases, a non-integer RHS value can be rounded-off (downwards or upwards depending on the constraint type: $\leq, \geq$) so that the associated constraint will be tighter.

**Example**

- Consider the constraint
  
  $$3x_1 + 4x_2 + 5x_3 \leq 36.78$$
  
  If all the variables are required to be integers, then this constraint can be replaced by
  
  $$3x_1 + 4x_2 + 5x_3 \leq 36$$
  
  Similarly, the constraint
  
  $$3x_1 + 4x_2 + 5x_3 \geq 36.78$$
  
  Can be replaced by
  
  $$3x_1 + 4x_2 + 5x_3 \geq 37$$

**The basic Idea**  
(*for equality constraints*)

- If the RHS value (RHSV) is not an integer, rewrite it as
  
  $$\text{RHSV} = I + f$$
  
  where $I$ is a non-negative integer and $f$ is a fraction. Note that $I=\text{Integer}[\text{RHSV}]$.
- If all the coefficients of the constraint are integers, then we can drop the fractional part of the RHS, namely $f$, and replace RHSV by $I$.  

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*How about = constraints?*
Example

\[3x_1 + 4x_2 + 5x_3 = 36.78\]

Thus, we have

\[\text{RHSV} = 36.78\]

\[\text{I} = \text{Integer}[\text{RHSV}] = 36\]

\[f = \text{RHSV} - \text{I} = 0.78\]

The modified (tighter) constraint is then

\[3x_1 + 4x_2 + 5x_3 = \text{I} = 36\]

Difficulty

- What should we do if (when using the simplex method) the coefficients of the constraint in question are not integers?

- Example:

  \[x_1 - 1.25x_2 + 0.25x_3 = 3.75\]

Remedy

1. Express each coefficient as the sum of an integer value and a non-negative fraction, that is

   \[C = I + f, \quad 0 \leq f < 1, \quad I = \text{integer}\]

Example:

\[-1.25 = -2 + 0.75 \quad (I = -2, \ f = 0.75)\]

\[1.25 = 1 + 0.25 \quad (I = 1, \ f = 0.25)\]

\[1 = 1 + 0 \quad (I = 1, \ f = 0)\]

Remedy

2. Then, move the fractional parts of the coefficients to the right hand side of the constraint:

- Example:

  \[x_1 - 1.25x_2 + 0.25x_3 = 3.75\]

  \[x_1 - 2x_2 + 0.75x_2 + 0x_3 + 0.25x_3 = 3.75\]

  \[x_1 - 2x_2 + 0x_3 = 3.75 - 0.75x_2 - 0.25x_3\]
Remedy

3. We then write the RHS as the sum of an integer value and one or more fractions: Example
\[ x_1 - 2x_2 + 0x_3 = 3.75 - 0.75x_2 - 0.25x_3 \]
\[ x_1 - 2x_2 + 0x_3 = [3 + 0.75] - 0.75x_2 - 0.25x_3 \]
\[ x_1 - 2x_2 + 0x_3 = 3 + 0.75 - 0.75x_2 - 0.25x_3 \]

Observe that
any optimal solution to the LP relaxation does not satisfy
\[ 0.75 - 0.75x_2 - 0.25x_3 <= 0 \]

(why?) whereas
any feasible solution to the IP problem does satisfy this
constraint. (why?)

Explanation

1. Why any feasible solution to the IP problem satisfies the cut?
From the final Simplex Tableau
\[ x_1 - 2x_2 + 0x_3 = 3 + 0.75 - 0.75x_2 - 0.25x_3 \]
For integer non-negative variables,
\[ 0.75 - 0.75x_2 - 0.25x_3 <= 0.75 < 1 \]

Hence, the LHS of (*) must be an integer <= 3.
This implies that any feasible solution to the IP problem satisfies the cut
\[ 0.75 - 0.75x_2 - 0.25x_3 <= 0 \]

Explanation

2. Why the optimal solution to the LP relaxation problem does not satisfy the cut?
From the final Simplex Tableau
\[ x_1 - 2x_2 + 0x_3 = 3 + 0.75 - 0.75x_2 - 0.25x_3 \]
The variables \( x_2 \) and \( x_3 \) are non-basic, hence \( x_2 = x_3 = 0 \).
It is obvious then that the optimal solution to the LP relaxation cannot satisfy the cut
\[ 0.75 - 0.75x_2 - 0.25x_3 <= 0 \]

Observe that this would imply
\[ 0.75 <= 0 \]

Cut

• A cut is the constraint obtained by forcing the sum of the fractions on the RHS to be non-positive:
• Example:
\[ x_1 - 2x_2 + 0x_3 = 3 + 0.75 - 0.75x_2 - 0.25x_3 \]
produces the cut
\[ 0.75 - 0.75x_2 - 0.25x_3 <= 0 \]
**Procedure**

- We keep appending the cuts to the problem until the LP relaxation produces an IP solution or we conclude that the problem is not feasible.
- When we add a cut, we use the Dual Simplex method to take care of negative right hand sides.

**Logic**

- Any feasible point for the IP problem will satisfy the cut.
- The current optimal solution to the LP relaxation will not satisfy the cut.
- Thus, there will be no “cycles” and the procedure will terminate (eventually).

**Geometry**

The cut does not cut out any feasible IP solution

**Cutting Plane Algorithm**

- **Step 1**: Solve the LP relaxation of the IP problem. If the solution is integer, Stop! This solution is also optimal for the IP problem.
- **Step 2**: Select a constraint whose RHS value is not integer.
- **Step 3**: Generate the cut for this constraint and append it to the tableau.
- **Step 4**: Use the Dual Simplex method to solve the LP relaxation.
- **Step 5**: If the LP solution is integer, Stop. Other wise go to step 2.
Example
(Winston section 9)

\[
\begin{align*}
\text{max } x & = 8x_1 + 5x_2 \\
\text{s.t.} & \\
x_1 + x_2 & \leq 6 \\
9x_1 + 5x_2 & \leq 45 \\
x_1, x_2 & \geq 0; x_1, x_2 \text{ integer}
\end{align*}
\]

• Pure IP problem

Cutting Plane Algo

• Initial tableau of the Simplex Method

<table>
<thead>
<tr>
<th>BV</th>
<th>Eq #</th>
<th>X1</th>
<th>X2</th>
<th>X3</th>
<th>X4</th>
<th>RHS</th>
</tr>
</thead>
<tbody>
<tr>
<td>X3</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>6</td>
</tr>
<tr>
<td>X4</td>
<td>2</td>
<td>9</td>
<td>5</td>
<td>0</td>
<td>1</td>
<td>45</td>
</tr>
<tr>
<td>Z</td>
<td>Z</td>
<td>-8</td>
<td>-5</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

• Final tableau of the LP relaxation
• The solution to the LP relaxation is not integer, thus we have to generate a cut.

<table>
<thead>
<tr>
<th>BV</th>
<th>Eq #</th>
<th>X1</th>
<th>X2</th>
<th>X3</th>
<th>X4</th>
<th>RHS</th>
</tr>
</thead>
<tbody>
<tr>
<td>X2</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>9/4</td>
<td>-1/4</td>
<td>9/4</td>
</tr>
<tr>
<td>X1</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>-5/4</td>
<td>1/4</td>
<td>15/4</td>
</tr>
<tr>
<td>Z</td>
<td>Z</td>
<td>0</td>
<td>0</td>
<td>5/4</td>
<td>3/4</td>
<td>165/4</td>
</tr>
</tbody>
</table>

• We select a row whose RHS value is closet to 0.5, (there is a tie in our case), say row 2.
• We now generate the cut for the constraint represented by this row.

<table>
<thead>
<tr>
<th>BV</th>
<th>Eq #</th>
<th>X1</th>
<th>X2</th>
<th>X3</th>
<th>X4</th>
<th>RHS</th>
</tr>
</thead>
<tbody>
<tr>
<td>X2</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>9/4</td>
<td>-1/4</td>
<td>9/4</td>
</tr>
<tr>
<td>X1</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>-5/4</td>
<td>1/4</td>
<td>15/4</td>
</tr>
<tr>
<td>Z</td>
<td>Z</td>
<td>0</td>
<td>0</td>
<td>5/4</td>
<td>3/4</td>
<td>165/4</td>
</tr>
</tbody>
</table>
The constraint is
\[ x_1 + 0x_2 -(5/4)x_3 + (1/4)x_4 = 15/4 \]
\[ x_1 + 0x_2 -2x_3 +(3/4)x_3 + (1/4)x_4 = 3 + 3/4 \]
\[ x_1 + 0x_2 -2x_3 = 3 + 3/4 -(3/4)x_3 - (1/4)x_4 \]
The cut is
\[ 3/4 -(3/4)x_3 - (1/4)x_4 <= 0 \]
\[ -(3/4)x_3 - (1/4)x_4 <= -3/4 \]

There is a negative RHS value (row 3), so we use the Dual Simplex Method to handle it.
The Ratio Test select x3 to enter the basis.
We thus pivot on (row 3, column 3)

We thus add the constraint
\[ -(3/4)x_3 - (1/4)x_4 <= -3/4 \]
To the simplex tableau.
But to guarantee that the slack variable induced by this constraint is also integer, we rewrite the constraint as
\[ -3x_3 - x_4 <= -3 \]

This is an “optimal tableau” for the LP relaxation.
Because the optimal solution is integer, it is also optimal for the IP.
Note
(Winston, 9.3 P 8)

- When you create a new sub-problem, at least one of the optimal solutions to the new LP relaxation problem will make the new cut (\(x_j \leq k\) or \(x_j \geq k\)) binding.
- This fact can be used to speed up the solution process.
- Try to proof this result.

Comments

- A good heuristic for the selection of a constraint for the generation of a cut is to select the row whose RHS value is closed to 1/2 (Why?)
- The cuts that we studied are called Gomory’s Cut
- There are other types of cuts
- There are cut-and-branch methods