Unit Distance Graphs With Ambiguous Chromatic Number

Michael Payne

Department of Mathematics and Statistics
University of Melbourne

Monday 7 December
Outline

1. Background

2. Construction of examples

3. Colourings under different axioms

Chromatic Number of the Plane

Problem

What is the minimum number $\chi(\mathbb{R}^2)$ of colours required to colour $\mathbb{R}^2$ so that no two points distance 1 apart receive the same colour?

It is known that $4 \leq \chi(\mathbb{R}^2) \leq 7$. 
Previous Results

**Theorem [Falconer, 1981]**
If we demand that the colour sets be Lebesgue measurable then
\[ n + 3 \leq \chi_m(\mathbb{R}^n). \]

**Theorem [Woodall, 1973]**
The rational plane can be 2-coloured, hence \( \chi(\mathbb{Q}^2) = 2 \).
Set Theory

- Completely general colourings can be done in $ZFC$.
- Restriction to measurable colourings is compatible with restricting the Axiom of Choice to countable collections and imposing $LM$ : \textit{All sets in $\mathbb{R}^n$ are Lebesgue measurable.}

Previous ambiguous examples:

- Székely, 1984] describes a unit distance graph on $S^1$ with $\chi = 2$ but $\chi_m = 3$.
- [Soifer and Shelah, 2004] describe non unit distance graphs on $\mathbb{R}^2$ and later $\mathbb{R}^n$. 

Michael Payne (University of Melbourne)
Set Theory

- Completely general colourings can be done in $ZFC$.
- Restriction to measurable colourings is compatible with restricting the Axiom of Choice to countable collections and imposing $LM: \text{All sets in } \mathbb{R}^n \text{ are Lebesgue measurable.}$

Definition: If the chromatic number depends on this choice (in other words if $\chi \neq \chi_m$) we say it is ambiguous.
Completely general colourings can be done in $ZFC$.

Restriction to measurable colourings is compatible with restricting the Axiom of Choice to countable collections and imposing $LM: \text{All sets in } \mathbb{R}^n \text{ are Lebesgue measurable.}$

Definition: If the chromatic number depends on this choice (in other words if $\chi \neq \chi_m$) we say it is ambiguous.

Previous ambiguous examples:

- [Székely, 1984] describes a unit distance graph on $S^1$ with $\chi = 2$ but $\chi_m = 3$.
- [Soifer and Shelah, 2004] describe non unit distance graphs on $\mathbb{R}^2$ and later $\mathbb{R}^n$. 
For any field $\mathbb{Q} \subset K \subset \mathbb{R}$ the graph $T_{K^n}$ is defined by translating the unit distance graph $K^n$ everywhere in $\mathbb{R}^n$.

- $V = \mathbb{R}^n$
- $E = \{\{p_1, p_2\} : p_1 - p_2$ is a unit vector in $K^n\}$
We can show that \( \chi(T_{Kn}) = \chi(K^n) \).

- Each coset in \( \mathbb{R}^n/K^n \) is disconnected from the others, so can be coloured like \( K^n \).
- For \( K \) countable we have to choose uncountably many representatives, one for each coset.
Measurable Colourings

The graphs $T_{K^n}$ have two useful properties:

- The neighbourhood of each vertex is dense in $S^{n-1}$.
- Edge set $E$ is invariant under arbitrary translations.
Measurable Colourings

The graphs $T_{K^n}$ have two useful properties:

- The neighbourhood of each vertex is dense in $S^{n-1}$.
- Edge set $E$ is invariant under arbitrary translations.

Definition: For a measurable set $S$ in $\mathbb{R}^n$ we define the essential part $\tilde{S}$ of $S$ to be the set of points where $S$ has Lebesgue density 1.
Measurable Colourings

The graphs $T_{K^n}$ have two useful properties:

- The neighbourhood of each vertex is dense in $S^{n-1}$.
- Edge set $E$ is invariant under arbitrary translations.

Definition: For a measurable set $S$ in $\mathbb{R}^n$ we define the essential part $\tilde{S}$ of $S$ to be the set of points where $S$ has Lebesgue density 1.

**Lemma**

If $S$ is Lebesgue measurable and $\tilde{S}$ realises distance 1, then $T_{K^n}$ has an edge with both vertices in $S$. 
Measurable Colourings

The graphs $T_{K^n}$ have two useful properties:

- The neighbourhood of each vertex is dense in $S^{n-1}$.
- Edge set $E$ is invariant under arbitrary translations.

Definition: For a measurable set $S$ in $\mathbb{R}^n$ we define the essential part $\tilde{S}$ of $S$ to be the set of points where $S$ has Lebesgue density 1.

Lemma

If $S$ is Lebesgue measurable and $\tilde{S}$ realises distance 1, then $T_{K^n}$ has an edge with both vertices in $S$.

- Choose a unit vector $v$ in $K^n$ very close to $p_1 - p_2$.
- There are neighbourhoods of $p_1$ and $p_2$ on which $S$ has large measure.
- Consider translates of $v$ in these neighbourhoods.
Bounds

**Theorem**

\[ \chi_m(T_{K^n}) \] is at least \( \chi(\mathbb{R}^n) \).

- Consider a colouring by \( \chi(\mathbb{R}^n) - 1 \) measurable sets \( \{S_i\} \).
- \( \chi(\mathbb{R}^n) \) is attained on a finite unit distance graph.
- This graph can be placed with each vertex in \( \tilde{S}_i \) for some \( i \).
- By the lemma \( T_{K^n} \) has a monochromatic edge.
Bounds

Theorem

$\chi_m(T_{Kn})$ is at least $\chi(\mathbb{R}^n)$.

- Consider a colouring by $\chi(\mathbb{R}^n) - 1$ measurable sets $\{S_i\}$.
- $\chi(\mathbb{R}^n)$ is attained on a finite unit distance graph.
- This graph can be placed with each vertex in $\tilde{S}_i$ for some $i$.
- By the lemma $T_{Kn}$ has a monochromatic edge.

Theorem

It is possible to adapt Falconer’s proof to show that $n + 3 \leq \chi_m(T_{Kn})$. 
Ambiguity

Corollary

Suppose $\chi(K^n) < n + 3$ or $\chi(\mathbb{R}^n)$, then $T_{K^n}$ has ambiguous chromatic number.

- Our original example is $T_{Q^2}$. Here $\chi = 2$ and $\chi_m \geq 5$.
- For $T_{Q^3}$ we have $\chi = 2$ and $\chi_m \geq 6$ [Benda and Perles, 2000].
- For $T_{Q^4}$ we have $\chi = 4$ and $\chi_m \geq 7$ [Benda and Perles, 2000].
- For $K = Q[\sqrt{3}]$, $n = 2$ we have $\chi = 3$ while $\chi_m \geq 5$ [Fischer, 1990].
- ...
References

Colorings of metric spaces.

The realization of distances in measurable subsets covering $\mathbb{R}^n$.

Additive $K$-colorable extensions of the rational plane.

Axiom of choice and chromatic number: examples on the plane.

Measurable chromatic number of geometric graphs and sets without some distances in Euclidean space.

Distances realized by sets covering the plane.