On the connectivity of Visibility Graphs

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Def: The visibility graph $V(P)$ of a finite set $P \subseteq \mathbb{R}^2$ has $P$ as its vertices and two points are adjacent if they ‘see’ each other, i.e., no other point of $P$ lies on the line segment between them.
Visibility graphs are useful: eg. Szekely's proof of Szemerédi-Trotter Theorem.

'Apply crossing lemma to vis. graph.'

Clique number vs. chromatic number:

3 \implies 3 \quad [k\alpha z, Poi, Wood]

6 \rightarrow \text{unbounded.} \quad [Pfender]

Big lie - big clique conjecture: [k, p, w]

Every sufficiently large point set in $\mathbb{R}^2$ contains either $t$ collinear points or a clique of size $k$.

(True for $k \leq 5$ and $k=6, t=3$.)


Edge Connectivity

Visibility graphs have diameter \( \leq 2 \).

Diameter 2 graphs have \( \lambda = 5 \) [Plesnik, 75]

In visibility graphs, edge cuts of size 5 only appear around a vertex.

Diameter 2 graphs with 5-cuts not around a vertex look like this:
Vertex Connectivity

There are visibility graphs with \( k \leq 5 \).

\[
\begin{align*}
K &= 2n + 1 \\
K &\approx \frac{2S}{3} \\
S &= 3n + 1 \\
(n=4)
\end{align*}
\]

**Def.** The *visibility graph* \( B(A,B) \) of two disjoint point sets \( A \) and \( B \) is the bipartite subgraph of \( V(A \cup B) \) induced by \( A \) and \( B \).
Thm: Let $A$ and $B$ be disjoint sets such that $A \cup B$ has at most $l$ points on a line. Then $B(A,B)$ has a plane subgraph with at least $\frac{n-1}{l-1}$ edges. ($n = |A \cup B|$)

Cor: If $P \subseteq \mathbb{R}^2$ with $|P| = n$ and at most $l$ points on a line, then $V(P)$ is $\frac{n-1}{l-1}$-connected.

Idea of proof: (of Theorem)
Then: Let $A$ and $B$ be disjoint sets of $n$ points such that $A \cup B$ is not on a line. Then $B(A,B)$ has a plane subgraph with at least $n+1$ edges.

Cor: If $P \subseteq \mathbb{R}^2$ does not lie on a line, then $V(P)$ has $K \geq \frac{5}{2} + 1$.

If $C$ separates $A$ from $B$ then
$$g \leq |A| + |C| - 1 \text{ and } |C| \geq |A| + 1.$$
**Theorem:** Let $A$ and $B$ be disjoint sets of $n$ points such that $A \cup B$ is not on a line. Then $B(A,B)$ has a plane subgraph with at least $n+1$ edges.

**Outline of Proof:** Induction on $n$. (Base case $n=2$).

**Case (i):** There is a line $L$ containing $n$ points.

**Use:** Lemma: Let $A'$ lie on a line $L'$ and $B'$ have no points on $L'$ with $|A'| \geq |B'|$. Then $B(A',B')$ has a plane subgraph with $|A'| + |B'| - 1$ edges.

**Set:** $L' = L$, $A' = A \cap L \geq B \cap L$, $B' = B \setminus L$.

We get $|A'| + |B'| - 1 \geq n - 1$ edges.

**Need 2 more:**

- One along $L$.

- $(A \cap L = B \cap L)$ can add one far away from $L$. 


Then: Let $A$ and $B$ be disjoint sets of $n$ points such that $A \cup B$ is not on a line. Then $B(A,B)$ has a planar subgraph with at least $n+1$ edges.

Case (ii) $\exists$ such a line. Apply the Ham Sandwich Theorem to find a line $h$ with at most half of each set on each side. Assign points on $h$ as follows: so that each side gets $\lceil \frac{n}{2} \rceil$ of each set. Applying induction on both sides gives $n+2$ edges. We need only delete one to avoid overlaps on $h$. 
Thm: For $t = 4$, $k \geq \frac{2}{3} s$.

The proof requires the following interesting lemma:

Lem: Let $G_1, G_2$ be properly coloured plane straight line graph drawings, separated by a line. Then a non-crossing properly coloured edge can be added between them.

Example:

Conj: $k \geq \frac{2}{3} s$ always.
**Connectedness of Bivisibility Graphs**

**Lem:** A ∪ B not on a line. T a triangle with vertices a ∈ A, b ∈ B, c ∈ A ∪ B. Then a or b has a neighbour in B(A, B) in T \ {a, b}.

**Thm:** B(A, B) not on a line. Then B(A, B) has at most one component that is not an isolated vertex.
Thm: \( B(A,B) \) not on a line. Then \( B(A,B) \) has at most one component that is not an isolated vertex.

Pf: 1. Suppose \( \exists \) two components with an edge.
   - Choose \( ab, a'b' \) so that \( \text{conv}(\{a, b, a', b'\}) =: C \) is minimal.
   - If \( a, b, a', b' \) are on a line, use closest pt.
   - Otherwise wlog \( a, b \) are vertices of \( C \).
   - Apply Lemma (with \( C = a' \) or \( b' \)).
   - If neighbour is not \( c \) then \( C \) was not minimal.