2004 Schools Maths Olympics
Solutions

Melbourne University Mathematics and Statistics Society

Discovery Day, Sunday 22 August 2004

Question 1

Bobbi and Chris were walking up the stairs of a tower. Bobbi was constantly 52 steps ahead of Chris. When Bobbi was halfway up the stairs, she said to Chris, "When I've reached the top, you'll be three times as far as you are now." What is the number of stairs in the tower?

Solution 1

Let x be the total number of stairs, and let y be the number that Chris has climbed when Bobbi is halfway. Then $\frac{x}{2} - y = 52$ and $y + \frac{x}{2} = 3y$. Eliminating y gives $x = 208$.

Question 2

In the lead-up to an election, the truthfulness of two leading politicians is known to follow the following pattern: John lies on Monday, Tuesday and Wednesday, and tells the truth on all other days. Mark, on the other hand, lies on Thursday, Friday and Saturday, and tells the truth on all other days. One day they both said "Yesterday was one of my lying days". On which day of the week did they say this?

Solution 2

None of the lying days coincide, so at least one of them is telling the truth. They cannot both be telling the truth since that would mean yesterday is a lying day for both of them. Thus, exactly one of them is lying, which means one of them is telling the truth today and therefore lying yesterday, and the other is lying today and therefore telling the truth yesterday. By inspection, the only day which is a changeover between lying and truthtelling for both politicians is Thursday.

Question 3

A square is cut along two lines parallel to a side to form three identical rectangles . If the perimeter of each rectangle is 24, then what is the area of the original square?

Solution 3

Let the side of the square be $3x$. Then the rectangles will be x by $3x$. Since they have perimeter 24, $x = 3$. Then the area of the square is $(3x)^2 = 81$.

Question 4

Express

$$\frac{2^{12}+2^{11}+2^{10}}{2^9+2^8+2^7}$$

in simplest form.

Solution 4

$\frac{2^{12}+2^{11}+2^{10}}{2^9+2^8+2^7} = \frac{2^3(2^9+2^8+2^7)}{2^9+2^8+2^7} = 2^3 = 8.$

Alternatively, cancelling a factor of 2^7 on top and bottom gives $\frac{2^5+2^4+2^3}{2^2+2+1}$. The numbers aren't very large now, so evaluating gives $\frac{32+16+8}{4+2+1} = \frac{56}{7} = 8.$

Question 5

A circular table has exactly 60 chairs around it. There are N people seated at this table in such a way that the next person to be seated must sit next to someone else. What is the smallest possible value of N?

Solution 5

If there is a gap of 3 anywhere, then we can let the next person sit in the middle of that gap, so every 3 seats must have at least one person, giving 20 as a lower bound. By putting 20 people evenly around the table with a space of 2 seats in between each pair, it is easy to see that it works. Thus the minimum value of N is 20.

Question 6

In ΔABC, $AB = 5$, $BC = 12$ and $\angle ABC = 90°$. Arcs of circles are drawn, one with centre A and radius 5, the other with centre C and radius 12. The two circles intersect segment AC at M and N respectively. What is the length of MN?

Solution 6

By Pythagoras, $AC = 13$. Also, $AM = 5$ and $CN = 12$, since they lie on circles of that radii at the respective centres. Hence $MN = AM + CN - AC = 4$.

Question 7

If we add 329 to the three digit number $2x4$, we get $5y3$. If $5y3$ is divisible by three, what is the greatest possible value of x?

Solution 7

To be divisible by 3, the sum of digits of $5y3$ must be divisible by 3, so $y = 1$, 4 or 7. Now consider the sum $329 + 2x4$. $9 + 4 = 13$, so there is a 1 carried over to the tens digits. $3 + 2 = 5$ so the sum of the tens digits and the carried over 1 cannot be greater than 10. This gives $2 + x + 1 = y$, so taking the maximum value of $y = 7$, the maximum value of x is 4.

Question 8

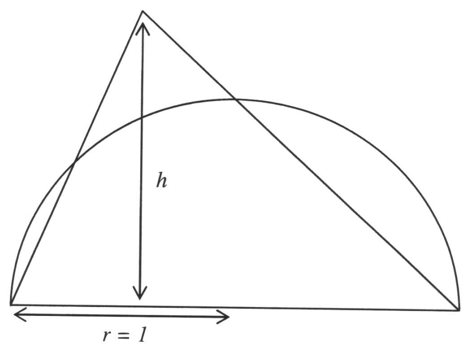
An urn is filled with coins and beads, all of which are either silver or gold. Twenty percent of the objects in the urn are beads. Forty percent of the coins in the urn are silver. What percentage of objects in the urn are gold coins?

Solution 8

$100\% - 20\% = 80\%$ are coins, and $100\% - 40\% = 60\%$ of coins are gold. Multiplying gives 48% gold coins.

Question 9

Consider the triangle and semicircle shown. The area of that part of the triangle lying outside the semicircle is equal to the sum of the areas of those parts of the semicircle lying outside the triangle. If the radius of the semicircle is 1, find the height of the triangle, h.



Solution 9

The area of the semicircle is $\frac{\pi}{2}$, half the area of a circle of radius 1. From the information in the question, we can conclude that the areas of the triangle and semicircle are equal. Half the base of the triangle is 1. Since area $= \frac{1}{2} \times \text{base} \times h$, then $h = \frac{\pi}{2}$.

Question 10

Let $t_1 = 22$. To obtain t_{n+1}, we square the sum of the digits of t_n.
For example, $t_2 = (2+2)^2 = 16$. Find t_{2004}.

Solution 10

By inspection, $t_3 = 49$, $t_4 = 169$, $t_5 = 256$, $t_6 = 169$. Since each term is based entirely on the previous term, any repeated value means the sequence must cycle. The period is equal to the difference between the nearest equal terms, here t_6 and t_4, and so the period of the cycle is 2. Then, each even subscripted term must be 169, and each odd subscripted term must be 256, from t_4 onwards. Hence $t_{2004} = 169$.

Question 11

Norm and Denise each wish to buy an ice cream which costs a whole number of dollars. However Norm needs seven more dollars to buy an ice cream, while Denise needs one more dollar. They decide to buy only one ice cream together, but discover that they do not have enough money. How much does one ice cream cost (in dollars)?

Solution 11

Suppose an ice cream costs x, Norm has n dollars and Denise has d dollars. Then $x-n=7$, $x-d=1$, $n+d<x$. Adding these equations together gives $2x<8+x$, so $x<8$. Now if $x<7$, and $x-n=7$, then $n<0$, which is impossible if we assume Norm does not have a negative amount of money. Since the cost of an ice cream is smaller than 8 but not smaller than 7, and a whole number, it must be 7 dollars.

Question 12

In how many ways can you colour three edges of a cube green so that each face has a green edge?

Solution 12

Each green edge is on two faces, so if each face of each green edge is unique, then there are 6 faces with green edges. We thus see that no face can have more than one green edge, or there would be less than 6 faces with green edges. The problem then becomes equivalent to the number of ways of finding three pairs of adjacent faces. The top face has 4 choices of an adjacent face, and having selected one, without loss of generality suppose it is the front face. Then the right face has two choices, the back face or the bottom face, and the there is only one choice for the last pair. Hence there are $4\times 2=8$ possibilities.

Question 13

Express the value of this sum in simplest form:

$$\frac{1}{2\lfloor\sqrt{1}\rfloor+1}+\frac{1}{2\lfloor\sqrt{2}\rfloor+1}+\ldots+\frac{1}{2\lfloor\sqrt{100}\rfloor+1}$$

(where $\lfloor x\rfloor$ is the greatest integer less than or equal to x)

Solution 13

Note that, removing the last term, the series can be broken up into segments consisting of sums of equal terms. Note that for a positive integer n, $(n+1)^2-n^2=2n+1$, so the number of integers at least n^2 but smaller than $(n+1)^2$ is $2n+1$. In other words, there are $2n+1$ integers x where $\lfloor\sqrt{x}\rfloor=n$. But then, the series just reduces to a sum of $(2n+1)\times\frac{1}{2n+1}=1$, and there are 9 of them from 1 to 99 ($\lfloor\sqrt{99}\rfloor=9$). So the sum of the first 99 terms is 9. Then we only have to add the last term, $\frac{1}{21}$, to obtain the sum of the series as $\frac{190}{21}$.

Question 14

If $t_1 = 1$ and $t_{n+1} = \frac{t_n}{1+t_n}$ for all positive integers n, what is the value of t_{2004}?

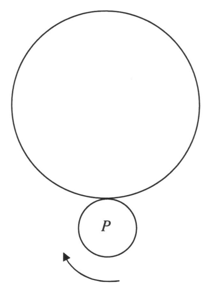
Solution 14

By inspection, $t_1 = 1$, $t_2 = \frac{1}{2}$, $t_3 = \frac{1}{3}$, and at this point one might guess $t_n = \frac{1}{n}$. If not, then calculating the next few terms should make it very obvious. It is in fact true, so $t_{2004} = \frac{1}{2004}$ is the correct answer. It can be quite easily proved by induction although it would be a silly thing to do when only the answer is required.

Unnecessary proof by induction: it is obviously true for t_1 so assume it is true for t_n. Then $t_{n+1} = \frac{t_n}{1+t_n} = \frac{1}{n} \times \frac{1}{1+\frac{1}{n}} = \frac{1}{n} \times \frac{1}{\frac{n+1}{n}} = \frac{1}{n+1}$, so $t_n = \frac{1}{n}$ is true for all n.

Question 15

The diameter of the small circle is a quarter of the diameter of the large circle. If the small circle rolls (without slipping) around the outside of the large circle, how many full rotations will the letter P (on the small circle) undergo by the time the small circle returns to its initial position?



Solution 15

Note that the ratio of diameters is equal to the ratio of circumferences, so the large circle has four times the circumference of the small circle. Let Q be the point on the small circle which is initially touching the large circle. At each of a quarter, a half, three quarters and the full way around the large circle, Q will be touching the large circle, after the small circle has rotated all the way around exactly once. However, the orientation of Q with respect to the centre of the small circle also changes, that is, when the small circle does a "full revolution", the point Q has done $1\frac{1}{4}$ revolutions. Thus after 4 revolutions to get back to where it started, the point Q, and hence the letter P, rotates 5 times.

Question 16

The points A, B, C, and D lie on a circle with centre O and radius r. $AB = 6$, $CD = 8$ and $BC = DA = \sqrt{2}r$. Find r.

Solution 16

$OA = OB = OC = OD = r$. By the converse of Pythagoras, $\angle AOD = \angle BOC = 90°$. Notice that we can rearrange the four triangles ABO, BCO, CDO, DAO to form a new quadrilateral $WXYZ$ inside the same circle of radius r with centre O where $WX = XY = \sqrt{2}r$ and $YZ = 6$ and $ZW = 8$. Since $\angle WOY = 180°$, $WY = 2r$ is a diameter of the circle and so $\angle WXY = \angle WZY = 90°$. Then by Pythagoras in ΔWYZ, $2r = WY = 10$ and so $r = 5$.

Question 17

Suppose

$$f\left(\frac{1}{x}\right) + \frac{1}{x}f(-x) = 2x$$

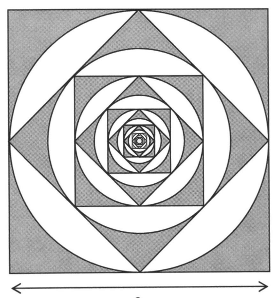
where f is a function defined for all non-zero real values of x. What is the value of $f(2)$?

Solution 17

Let $x = \frac{1}{2}$, then $f(2) + 2f(-\frac{1}{2}) = 1$. Let $x = -2$, then $f(-\frac{1}{2}) - \frac{1}{2}f(2) = -4$. Taking twice the second equation and subtracting the first equation yields $-2f(2) = -9$, giving $f(2) = \frac{9}{2}$.

Question 18

Find the shaded area.



Solution 18

Let the shaded area be x. The shaded area between the largest square and circle is $2^2 - \pi \cdot 1^2 = 4 - \pi$. Note that the side length of each square is $\frac{1}{\sqrt{2}}$ the side length of the next largest square. So the shaded area between a circle and square will be $\left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2}$ the shaded area between the next largest circle and square. So

$$x = (4-\pi) + \frac{1}{2}(4-\pi) + \left(\frac{1}{2}\right)^2(4-\pi) + ... = (4-\pi)(1 + \frac{1}{2} + \frac{1}{4} + ...) = 2(4-\pi)$$

Alternatively, by the symmetry of the figure, the ratio of shaded and non-shaded areas in the largest diamond inside the square is equal to that ratio for the entire square. Note that the diamond has half the area of the square, so it has half the shaded area, or $\frac{1}{2}x$. So,

$$\text{shaded inside diamond} + \text{shaded outside diamond} = \text{shaded for entire square}$$

giving

$$\begin{aligned} \tfrac{1}{2}x + (4-\pi) &= x \\ \Rightarrow \qquad x &= 2(4-\pi) \end{aligned}$$

Question 19

You are asked to form a committee of six people from a group of 10 couples. There are five 'happy' couples and five 'grumpy' couples. A member of a 'happy' couple will only serve on the committee if their beloved is also on the committee, while a member of a 'grumpy' couple refuses outright to work on the committee if their partner is on the committee. In how many ways can you form the committee?

Solution 19

The number of happy people on the committee must be even; note that 0 is not an option because 6 grumpy people together is impossible with only 5 grumpy couples, so there must be either 2, 4 or 6 happy people on the committee.

2 happy people: select 1 happy couple from 5, 4 grumpy couples from 5, 1 grumpy person from each included grumpy couple, so $5 \times 5 \times 2^4 = 400$ possibilities.

4 happy people: select 2 happy couples from 5, 2 grumpy couples from 5, 1 grumpy person from each included grumpy couple, so $10 \times 10 \times 2^2 = 400$ possibilities.

6 happy people: select 3 happy couples from 5, so 10 possibilities.

Adding them up gives a total of 810 possible committees.

Question 20

Daniel and Joanna, whilst journeying through outer space, came across a spaceship wreck, and with it an injured yet amiable alien who had lost his fingers. The pair of travellers were keen to find the alien's fingers for him, but did not know how many to look for. Just prior to crashing his spaceship, the alien was, naturally, doing some quick calculations on a piece of paper which Daniel rescued from the wreckage. It read:

$$\begin{aligned} 5x^2 - 50x + 125 &= 0 \\ \therefore x = 5 \quad \mathit{or} \quad x &= 8 \end{aligned}$$

Joanna quickly realized that the number base in which the alien was working was probably equal to the number of fingers he ought to have. How many alien-fingers were Joanna and Daniel looking for?

Solution 20

Let b be the base in question, and note that "50" means $5b$ and "125" means b^2+2b+5. The equation then becomes $b^2 + b(2 - 5x) + 5(x^2 + 1) = 0$. Substitute in the known roots $x = 5$ and $x = 8$ to get $b^2 - 23b + 130 = (b-10)(b-13) = 0$ and $b^2 - 38b + 325 = (b-13)(b-25) = 0$. The only solution consistent with both of these is $b = 13$.

Question 21

You are given a 5×5 grid and you colour in some of its squares. A colouring is called *permissible* if for any coloured-in square, either its whole column is coloured in or its whole row is coloured in. How many *permissible* colourings are there?

Solution 21

Since each coloured square must have a coloured column or row, the problem is equivalent to finding the number of different colourings feasible by colouring when one can only colour in an entire row or an entire column at a time. The number of different colourings of rows is $2^5 = 32$, since each row can be either coloured or not coloured. If all the rows are coloured, then any colouring of the columns will make no difference, so there is only one possibility here. For the remaining 31 cases, all possible colourings of columns will be unique except again for the case when all columns are coloured. Thus, there are $1 + 31^2 = 962$ possible colourings.

Question 22

The polynomial $p(x) = ax^4 + bx^3 + cx^2 + bx + a$ has exactly two real roots. If $x = 29$ is one solution of $p(x) = 0$, what is the other solution of $p(x) = 0$?

Solution 22

Knowing it has exactly two real roots, and given the symmetry of the polynomial, the obvious solution to try is equating the polynomial to $a(x-p)^2(x-q)^2$ where p and q are the roots. Multiplying out results in $p^2q^2 = 1$, so $q = \frac{1}{p}$, and so if $p = 29$ is a root, $q = \frac{1}{29}$ is the other root. Note that the question implies that this other root is uniquely determined, and thus this root satisfies all possible forms of the polynomial; a proof by considering each of these forms is quite possible but entirely a waste of time.

Alternatively, dividing through by x^2 and factorising gives $a(x^2 + \frac{1}{x^2}) + b(x + \frac{1}{x}) + c = 0$, which is symmetric in x and $\frac{1}{x}$, showing that whenever x is a root, $\frac{1}{x}$ is also a root.

Question 23

Circles C_1 and C_2 intersect at points A and B. They have common tangents CD and EF, where C and E lie on C_1, and D and F lie on C_2. The line DA passes through the midpoint of segment CE. If $CE = 7$ and $AB = 12$, find the ratio of the radius of C_1 to the radius of C_2. Express your answer in simplest form.

Solution 23

Let P be the midpoint of CE, Q be the midpoint of AB and R be the midpoint of DF. By symmetry, PQR is a straight line, and is perpendicular to all of CE, AB and DF. Also, $PQ = QR$ (a fact which is well known enough not to be obscure but not quite fitting of the title "well known"). ΔPQB and ΔPRF are similar, so $\frac{FR}{BQ} = \frac{PR}{PQ} = 2$, so $FR = 12$. Construct lines from the tangent points to the centres of each circle, and observe that the ratio of the radii is equal to the ratio of the lines connecting the tangent points, namely CE and DF. Since $CE = 7$ and $DF = 24$, the ratio of the radii of C_1 and C_2 is $7 : 24$.

Proof of "fact": Let X be the intersection of AB and CD. $\angle XCA = \angle ABC$, from the alternate segment theorem. $\angle CXB$ is common, so ΔXBC and ΔXCA are similar, and hence $\frac{AX}{CX} = \frac{CX}{BX}$, and so $CX^2 = AX \cdot BX$. Similarly, $DX^2 = AX \cdot BX$ and so $CX = DX$. Since CE, AB and DF are parallel by symmetry, it follows that $PQ = QR$.

Question 24

Find the number of 4×4 arrays with entries from $\{1, 2, 3, 4\}$ such that the sum of each row is divisible by 4 and the sum of each column is divisible by 4.

Solution 24

Consider filling out the top 3×3 portion of the array, the number of ways of which is obviously 4^9. Then, the six remaining entries that are not the bottom right corner are determined uniquely, in order to make the row totals and column totals a multiple of 4. Pick the remaining entry to make the 4th column total a multiple of 4; there is again a unique way of doing this. Then, each column has sum a multiple of 4, so the entire array has sum a multiple of 4, and since the three top rows also sum to multiples of 4, then the sum of the fourth row must also be a multiple of 4. Thus, for each selection in the top left 3×3 portion of the array, there is exactly one way of filling out the rest of the array, so the total number of possible arrays is 4^9.

Question 25

In ΔABC, $AB = 360$, $BC = 507$ and $CA = 780$. M is the midpoint of AC, D is the point on AC such that BD bisects $\angle ABC$. F is the point on BC such that BD and DF are perpendicular. The lines FD and BM meet at E. What is the value of $\frac{DE}{EF}$?

Solution 25

Extend FD to meet BA at X. Applying Menelaus' Theorem to ΔBXF (there is no point providing a less technical solution as this question is close to impossible without prior knowledge of this theorem, or at least familiarity with its result), $AX \cdot DF \cdot BC = AB \cdot DX \cdot CF$, which rearranges to $CF = \frac{AX \cdot BC}{AB}$. $DF = DX$ and $BX = BF$ because BD bisects $\angle XBF$ and is perpendicular to XF. Then,

$$BF = BC - CF = BC - \frac{AX \cdot BC}{AB} = BC - \frac{(BX - AB)BC}{AB} = BC - \frac{(BF - AB)BC}{AB}$$

A bit of algebra yields $BF = 2\frac{BC \cdot AB}{AB + BC}$, and so $\frac{BF}{BC} = \frac{720}{867}$.

$\frac{AD}{CD} = \frac{AB}{BC}$ since BD is an angle bisector. To see this, let $\angle ABD = \angle CBD = x$, $\angle ADB = y$. Then

$$\text{Area}(\Delta ABD) = \frac{1}{2}AB \cdot BD \sin(x) = \frac{1}{2}BD \cdot AD \sin(y) \qquad \text{and}$$

$$\text{Area}(\Delta CBD) = \frac{1}{2}BC \cdot BD \sin(x) = \frac{1}{2}BD \cdot CD \sin(180 - y) = \frac{1}{2}BD \cdot CD \sin(y)$$

Divide one equation by the other to get the required result.

Returning to the problem, $\frac{AD}{CD} = \frac{AM - DM}{CM + DM} = \frac{390 - DM}{390 + DM}$ since M is the midpoint of AC. Since $\frac{AD}{CD} = \frac{AB}{BC} = \frac{360}{507}$, solving for DM yields $DM = \frac{147}{867} \times 390$, and $\frac{DM}{CM} = \frac{147}{867}$. Finally, apply Menelaus to ΔCDF to get $DE \cdot BF \cdot MC = EF \cdot BC \cdot DM$, so

$$\frac{DE}{EF} = \frac{BC}{BF} \times \frac{DM}{CM} = \frac{147}{720} = \frac{49}{240}$$

Note that the last question is designed to be fiendishly hard so that no one can complain about finishing all the questions and having nothing to do. It is not expected that anyone will solve it in the duration of the competition.