SMO 2005 Questions and Answers

1. One bottle contains one litre of a mixture that is $\frac{1}{3}$ orange juice and $\frac{2}{3}$ carrot juice. Another bottle contains two litres of a mixture that is $\frac{3}{4}$ orange juice and $\frac{1}{4}$ carrot juice. These two bottles are poured into an empty jug. Find the final ratio of orange juice to carrot juice in this jug. Express your answer in the simplest form $a : b$ where a and b are integers.
Answer: 11:7

2. The roots of the equation $x^2+4x-5=0$ are also the roots of the equation $2x^3+9x^2-6x-5=0$. What is the other root of the second equation?
Answer: -0.5

3. Two cars are initially 500 metres apart. They speed towards each other, one at 30m/s and the other at 20m/s. How far apart, in metres, are they one second before crashing?
Answer: 50

4. An island is inhabited only by knights and knaves. Knights always tell the truth, and knaves always lie. Andrew and James are inhabitants. Andrew says: 'James is a knave.' James says: 'Neither Andrew nor I are knaves.' List, respectively, what Andrew and James are.
Answer: Knight, Knave

5. A square box of side 5cm is leaning against a vertical wall with the bottom corner of the square 4cm from the wall. Find the height of the highest corner of the square in centimetres.
Answer: 7

6. Nick's watch is 10 minutes fast but he thinks it is 5 minutes slow. Michael's watch is 5 minutes slow but he thinks it is 10 minutes fast. Thara's watch is 5 minutes fast but he thinks it is 10 minutes slow. Han's watch is 10 minutes slow but he thinks it is 10 minutes fast. Using their watches, each of them leaves for university at what each believes is the time to catch the 8:30am tram from Flinders Street. Who will miss their tram?
Answer: Michael and Han

7. For integers a and b we define $a * b = a^b + b^a$. If $2 * x = 177$, find x.
Answer: 7

8. I have five matchsticks which have lengths 2, 3, 4, 5 and 6 cm respectively. How many different triangles can I make?
   Answer: 7

9. Maurice has a standard die with six sides. Julian has a four-sided die with the numbers 0, 2, 4, 6 printed on each of the four sides. Maurice and Julian roll their own die. What is the probability that Julian rolls a higher number than Maurice? Express your answer as a fraction in the simplest form.
   Answer: $\frac{3}{8}$

10. A ball with radius 17cm is floating so that the top of the ball is 2cm above the smooth surface of a swimming pool. What is the circumference, in centimetres, of the circle formed by the contact of the water surface with the ball? Give exact answers.
    Answer: 16π

11. In a cubic room, a spider is at one of the corners in the ceiling. A fly is on the floor and at the midpoint of an edge that is on the opposite side of the room to where the spider is. If the shortest distance that the spider has to crawl in order to reach the fly is $\sqrt{13}$ metres, what is the sidelength of the room in metres?
    Answer: 2

12. Let x and y be real numbers such that:

    $$x^2 + 2y^2 - 2xy + 2y + 1 = 0$$

    What is x?
    Answer: -1

13. Determine the smallest positive integer n such that $n^3 + 2n^2 = b$ where b is a square of an odd integer.
    Answer: 7

14. Let a, b and c be the digits of a three-digit number satisfying the equation $49a + 7b + c = 286$. What is $a + b + c$?
    Answer: 16

15. What is the smallest integer greater than zero that is divisible by 225 and, when written in decimal representation, contains only zeros and threes as digits?
    Answer: 33300

16. Two equilateral triangles of sidelength 1 are placed side by side to form a rhombus. Let the diagonals of this rhombus intersect at O. If the rhombus is rotated about O, what is the area which is always within the rhombus? Give exact answers.
    Answer: $\frac{3\pi}{16}$

17. Joanna and her grandmother have the same birthday. It is found that for six consecutive years, Joanna's age is a divisor of her grandmother's age. What is the age difference between Joanna and her grandmother in years?
Answer: 60

18. How many positive integers less than one million have all their digits different?
Answer: 168570

19. Let $x_1 = 1$, and for $n \geq 1$, $x_{n+1} = \begin{cases} x_n^2 + n & n \text{ even} \\ \lfloor \sqrt{x_n} \rfloor & n \text{ odd} \end{cases}$
What is x_{2005}?
Note that $\lfloor x \rfloor$ means the largest integer less than or equal to x.
Answer: 1004005

20. Sally began collecting calendars in 2005 and will do so every year until the time when every subsequent year can be served by at least one of the calendars that she has already collected. When is the last year in which she must collect a calendar?
Answer: 2032

21. A group of children share marbles from a bag. The first child takes one marble and a tenth of the remainder. The second child takes two marbles and a tenth of the remainder. The third child takes three marbles and a tenth of the remainder. And so on until the last child takes whatever is left. Knowing that all the children end up with the same number of marbles, how many children were there?
Answer: 9

22. Find the smallest n such that if $10^n = X \times Y$, then at least one of X or Y must contain the digit 0.
Answer: 8

23. Let $ABCD$ be a convex quadrilateral such that $AB = 8$, $BC = 6$ and $BD = 10$. Also, $\angle BAD = \angle CDA$ and $\angle ABD = \angle BCD$. What is CD?
Answer: $\frac{64}{5}$

24. Maurice and Geordie arrive at a cafe independently at random times between 9am and 10am and each stay for m minutes. What is m if there is a 51% chance that they are in the cafe together at some moment.
Answer: 18

25. Triangle ABC has $AB = AC$ and $\angle BAC \leq 90$. P lies on AC, and Q lies on AB such that $AP = PQ = QB = BC$. Find ratio of $\angle ACB$ and $\angle APQ$.
Answer: 7:4