University Maths Olympics 2006
Questions and Answers

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1. Daniel likes two things—kittens and prime numbers. One day, he decides to fill up his whiteboard with kittens and prime numbers. He begins by drawing five cute kittens side by side on the board, and then writes the following numbers under the kittens: 110437, 124297, 138157, 152017. Before he writes a number under the last kitten, however, he suddenly realises that he forgot to check whether the numbers he wrote down are prime. Feeling adventurous, he decides to test their primality by writing down all the numbers up to 152017 on the back of his whiteboard and crossing out every second number after 2, then every third number after 3, and so on until only the primes are left. Satisfied that the four numbers he wrote down are prime, and also pleased at the size of his whiteboard and the amount time he just wasted, Daniel returns to the front of his whiteboard and writes down a number under the last kitten. What number does he write down?

2. A survey of the population of Boganville finds that 60% are female, 70% have blue eyes and 80% have blonde hair. What is the smallest percentage of the population that is certain to be female with blue eyes and blonde hair?

3. Justin is reading a tourist guide listing some of the straight-line distances between five towns: Aberystwyth, Botcherby, Caarnduncan, Damnaglaur and Evercreech. They are:

   Aberystwyth to Botcherby: 102km
   Botcherby to Caarnduncan: 79km
   Caarnduncan to Damnaglaur: 231km
   Damnaglaur to Evercreech: 16km
   Evercreech to Aberystwyth: 34km

   If Justin is currently in Botcherby, how far is he from Evercreech?

4. Gareth's house number satisfies the following properties:

   If it is a multiple of 3, then it is between 50 and 59 inclusive.
   If it is not a multiple of 4, then it is between 60 and 69 inclusive.
   If it is not a multiple of 6, then it is between 70 and 79 inclusive.

   What is his house number?

5. The letters a, b, c, d, e, f and g each represent a distinct digit (number from 0 to 9 inclusive). The three products $a \times b \times c$, $c \times d \times e$ and $e \times f \times g$ are equal. What 3 digits are not used?

6. Take a regular pentagon of side length 5cm. Sit an equilateral triangle, also with side length 5cm, on its interior, and roll it so that it touches all the sides of the pentagon once before returning to its original position. How far has the corner that was initially not touching the pentagon travelled?

7. A farmer wants to weigh four sacks of grain, but due to new Australian Wheat Board regulations, he can only weigh two sacks at a time. He obtains the following measurements: 22kg, 25kg, 31kg, 31kg, 37kg and 40kg. How much did the heaviest sack weigh?

8. A calculator has 10 buttons in the following arrangement:

7	8	9
4	5	6
1	2	3
0		

   What is the largest number that can be typed only making moves from one button to another with knight's moves (ie, 2 squares in one direction followed by 1 square in a perpendicular direction), and never using the same digit twice?

9. Consider the following sequence:

$$\begin{aligned} x_n &= 7x_{n-1} \text{ for even positive integers } n \\ x_n &= 1 + x_{n-1} \text{ for odd positive integers } n \\ x_0 &= 0 \end{aligned}$$

   Find the last two digits of x_{2006}.

10. Hans bought 3 items at a supermarket. In total, they cost \$90.09. He also notices that if he had multiplied the costs together (in dollars), he would also get \$90.09. What was the most expensive item worth?

11. What is the side length of the largest equilateral triangle that can fit inside a square of area 1?

12. A large building has 64 rooms arranged in a $4 \times 4 \times 4$ lattice. Each room has doors to each of the adjacent rooms on the same floor, and stairs to the corresponding rooms in adjacent floors. If Thara has just entered the building and is in one corner of the building on the bottom floor, and wants to get to his bedroom in the opposite corner of the building on the top floor, in how many ways can he get there in the minimum number of moves?

13. Tom, Dick and Harry attend the same gym. One attends every 2 days, one every 3 days and one every 7 days. In a particular month of gym attendance, Tom's first visit was on a Monday, Dick's on a Wednesday and Harry's on a Friday. Also, on one day of this month, all three attended the gym together. Which day of the month was it?

14. ABC is a trangle of area 1. Points X, Y, Z lie on the extended lines BC, CA and AB respectively, such that $BX = 3BC$, $CY = 3CA$, $AZ = 3AB$ and the triangle XYZ contains the triangle ABC. What is the minimum possible area of triangle XYZ?

15. If the product of four consecutive positive integers is 90345024, what is their sum?

16. x is a two-digit number such that when x^2 and x^3 are written down, every digit from 0 to 9 occurs exactly once. What is x?

17. Triangle ABC is isoceles with $\angle ABC = \angle ACB = 80^\circ$. D lies on AC such that $\angle DBC = 60^\circ$ and E lies on AB such that $\angle ECB = 50^\circ$. Find $\angle EDB$.

18. Let A be the set of non-degenerate triangles with perimeter 2006 and integer side lengths (ignoring permutations), and B be the set of triangles with perimeter 2009 under the same conditions. How many more triangles are there in B than A?

19. Three sets of twin girls are married to three sets of twin boys such that each girl's sister is married to her husband's brother. One day, the six couples decide to play mixed doubles tennis. However, no girl wants to be paired with her husband, or her husband's brother. How many possible pairings are there?

20. John is a judge with a keen interest in numerology. He uses the code A=1, B=2, C=3, etc to encode his wife's name. He notices that the product of the letters of her name is the same as for his profession, JUDGE. Given that it has no letter in common with JUDGE, and that its letters are in alphabetical order if you swap two of them, what is her name?

21. For each integer x, the number $f(x)$ is given by the formula

$$f(x) = \begin{cases} x - 2006 \text{ for } x > 2060 \\ f(f(x + 2007)) \text{ for } x \leq 2060 \end{cases}$$

Find $f(0)$;

22. Consider the function

$$f(x) = \lfloor x \rfloor + \lfloor \frac{4x}{3} \rfloor + \lfloor \frac{3x}{2} \rfloor + \lfloor 2x \rfloor$$

where $\lfloor x \rfloor$ represents the largest integer less than or equal to x.

How many different values does f take when $0 \leq x \leq 2006$?

23. Given $x + y + z = 1$, $xy + yz + xz = 2$ and $x^5 + y^5 + z^5 = 5$, what is xyz?

24. A and C lie on a circle with centre O and radius $\sqrt{50}$. The point B inside the circle is such that $\angle ABC = 90°$, $AB = 6$ and $BC = 2$. Find OB.

25. If S is a set of points in the plane, a point x is said to belong to the environment of S if there exist points y and z in S with the property that the distance between x and y is less than the distance between y and z. If S is an isoceles right-angled triangle of area 1, what is the area of its environment?

Answers

1. 165877
2. 10%
3. 136km
4. 76
5. 0, 5, 7
6. 4π cm
7. 23kg
8. 8167294305
9. 99
10. \$77
11. $\sqrt{2}(\sqrt{3}-1)$ OR $2\sqrt{2-\sqrt{3}}$
12. 1680
13. 27
14. 19
15. 390
16. 69
17. 30°
18. 502
19. 80
20. FANNY
21. 55
22. 7356
23. $\frac{6}{5}$
24. $\sqrt{26}$
25. $3+\sqrt{3}+\frac{31}{6}\pi$