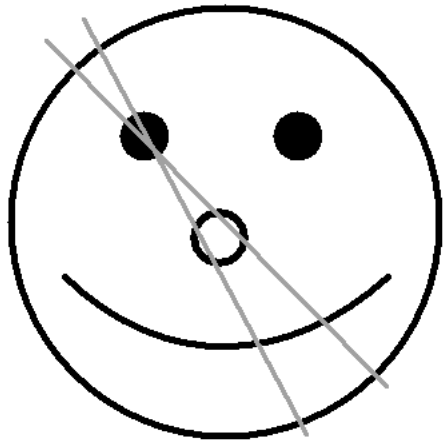
Schools Maths Olympics 2008: Solutions

1. Cuts crossing each other in a hollow region achieve nothing. Cuts crossing each other in a filled in region, such as an eye, add 1 to the number of pieces of icing.



   This construction gives the answer: $4+4+4+3+1=\mathbf{16}$.

   Comment: It was easy to get 15 by the cuts each crossing a different eye, and intersecting at the nose (the "obvious" way). Having guessed this incorrectly, 16 is a natural next guess.

2. $\frac{1}{8}+\frac{7}{12}=\frac{3+14}{24}=\frac{17}{24}$, so the midpoint is: $\mathbf{\frac{17}{48}}$.

3. $x\%$ is the same as $0.01x$, hence: $0.01x \times 12 = 0.12 \times x = \mathbf{7}$.

4. It was Sunday when this question was posed. 2008 leaves a remainder of 6 when divided by 7, so you may as well have 6 apples. This takes care of Sunday, Monday,…,Friday. Then on **Saturday** you'll have no apples left to eat.

   Comment: If you really don't know how to solve this, guessing is a perfectly good method (especially if you're close to the front of the room!).

5.
$$\frac{2+x}{3+x}=\frac{20+x}{23+x}$$

   Cross multiplying (equivalently, multiplying both sides by $(3+x)(23+x)$) we obtain:

$$\begin{aligned}(2+x)(23+x) &= (3+x)(20+x)\\ x^2+25x+46 &= x^2+23x+60\\ 2x &= 14\\ x &= \mathbf{7}\end{aligned}$$

6. We can take the given information and set up the following addition:

$$\begin{array}{r} \text{7abcd} \\ \underline{+\text{abcd7}} \\ 14000 \end{array}$$

From this, we deduce: $d = 3 \Rightarrow c = 6 \Rightarrow b = 3 \Rightarrow a = 6$; after verifying that it works, we know that the first number is **63637**.

7. There are $6 \times 6 = 36$ ways of eating one lolly from each bag. $3 \times 3 = 9$ of these ways don't involve eating snakes. Hence there are $36 - 9 =$ **27** ways of eating one lolly from each bag such that she eats at least one snake.

8. Let R be the radius of the pizza, then its circumference is $2\pi R$. Adding the perimeters of the 3 sectors, we get the circumference and 6 times the radius. Thus:

$$5 + 6 + 7 = 2\pi R + 6R = R(2\pi + 6)$$

$$\Rightarrow R = \tfrac{18}{2\pi+6} = \boldsymbol{\tfrac{9}{\pi+3}}.$$

9.

$$(a - b)^2 = a^2 + b^2 - 2ab = 4ab - 2ab = 2ab \quad (1)$$
$$(a + b)^2 = a^2 + b^2 + 2ab = 4ab + 2ab = 6ab \quad (2)$$

As $ab \neq 0$, we can take (1) $\div$ (2):

$$\frac{(a - b)^2}{(a + b)^2} = \frac{2ab}{6ab} = \frac{2}{6} = \boldsymbol{\frac{1}{3}}.$$

Comment: it is important to note that $ab \neq 0$. With this sort of problem, it is often not necessary to find explicit solutions for a and b.

10. Let the number be $10a + b$. Where a is the *tens* digit, and b is the *units* digit. Observe that a and b are integers from 1 to 9 (inclusive), as it's a 2-digit number. Then we have:

$$\begin{aligned} 10a + b &= 2ab \\ 2ab - 10a - b &= 0 \\ (2a - 1)(b - 5) &= 5. \end{aligned}$$

As $2a - 1$ is positive, so too must $b - 5$. Since 5 is prime, we have either:

$$2a - 1 = 1, \quad \text{and,} \quad b - 5 = 5, \text{ or}$$
$$2a - 1 = 5, \quad \text{and,} \quad b - 5 = 1$$

The former gives $b = 10$, which is impossible since b is a digit. Hence, $a = 3$ and $b = 6$. So, the number is **36**.

Comment: guessing might have been faster.

11. Some degree of trial and error is necessary for this question. First notice that $160/5 = 32$, $100/3 \approx 33$ and $240/7 \approx 34$. So Sam probably has a slight preference for steak, over sausages, and then lamb...but only slight! So it's probably more important to spend all of the \$53. Sam can buy 5 steaks and 6 sausages for $5 \times \$7 + 6 \times \$3 = \$53$, receiving meat weighing: $5 \times 240g + 6 \times 100g = 1200g + 600g = \mathbf{1800g}$.

12. Notice that for each A, there are 4 adjacent 'P's, and similarly, for each T, there are 4 adjacent 'H's. Also notice that we can go from either A to either one of the 'T's. Hence, there are $4 \times 2 \times 2 \times 4 = \mathbf{64}$ 'PATH's possible.

13. From the given information, we deduce that a and b are both negative. In particular, this means that $\sqrt{a^2} = -a$. Moreover, as $a = \frac{1}{b}$ and $b = \frac{1}{a}$, we have:

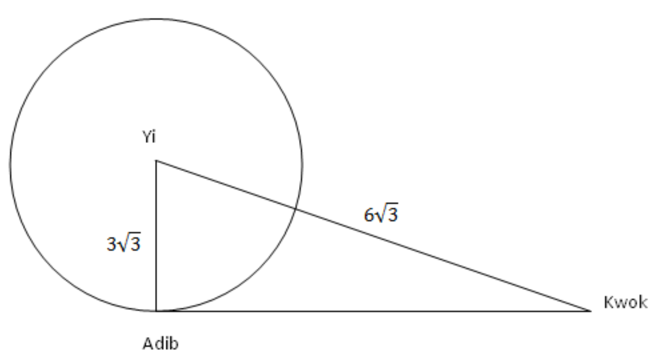
$$\sqrt{\frac{a}{b}} + \sqrt{\frac{b}{a}} = \sqrt{a^2} + \sqrt{b^2} = (-a) + (-b) = -(a+b) = \mathbf{3}.$$

Comment: remember to check that a and b are nonzero before asserting that $a = \frac{1}{b} \ldots$etc. Notice also that a^2 and b^2 are positive, regardless of the signs of a and b, and so the square root made sense.
Some students gave the answer -3, but remember that squares and square roots of real numbers are never negative, by definition! E.G.: $\sqrt{4} = 2 \neq -2$; although both 2 and -2 are solutions to the equation $x^2 = 4$. Or to put it in other notation, $\sqrt{x^2} = |x|$: the absolute value of x.

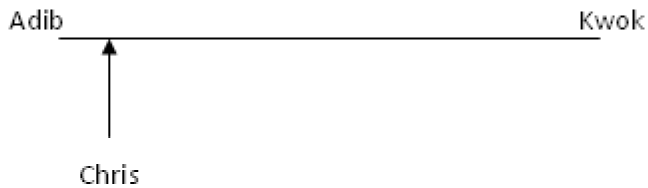
14. We can obtain 7 in the following ways:
$(1, 3, 3)$ with 3 possible orders,
$(2, 2, 3)$ with 3 possible orders,
$(1, 1, 5)$ with 3 possible orders,
and $(1, 2, 4)$ with $3! = 6$ possible orders.

The first 3 cases have a repeated number, so there are $3 + 3 + 3 = 9$ ways of achieving 7 with a repeated number. There are $3 + 3 + 3 + 6 = 15$ ways in total of achieving 7. So if the sum of 3 dice is 7, then the probability of two dice showing the same number is $\frac{9}{15} = \mathbf{\frac{3}{5}}$.

15. Adib must lie on a sphere, centred at Yi. For illustrative purposes, we draw a circle:



We maximise $\angle AKY$ by making AK tangent to Adib's orbit around Yi (The rigorous proof of this intuitive claim is left as an exercise: think of KY as a chord of the circumcircle of $\triangle AKY$). Now $\angle YAK = 90^o$, so by Pythagoras' Theorem, we get $\overline{AK} = 9 = 8+1$. We check that this event can occur:



We claim that $\angle AKY = \mathbf{30^o}$. This is immediate with trigonometry, or guessable otherwise. This can also be proven without trigonometry (Exercise! Let Y' be the reflection of Y across AK, and observe that $\triangle YKY'$ is equilateral).

Comment: this question perhaps contained too much information (more specifically, in-jokes for our amusement). The salient point was that $\angle AKY$ was maximised when AK was tangent to the circle.

16. We consider the numbers as 3-digit numbers, possibly beginning with 0 (E.G.: $0 = 000$ and $8 = 008$), and observe that this does not change whether or not the digits are all even. Moreover, we can include 000 - for convenience - even though it is not positive, because it does not affect the sum in any way.

    Notice that each of the 3 digits is 0, 2, 4, 6, or 8, and that there are no other restrictions. So, for instance, 0 occurs as the first digit in $5 \times 5 = 25$ of these numbers, since the second and third digits can be any even digits. Similarly, it occurs as the second digit in 25 of these numbers, and as the third digit in 25 of these numbers. The same can be said about 2, 4, 6 and 8. Now the sum of all of these numbers is:

$$(0+2+4+6+8)(1+10+100)\times 25 = \mathbf{55500}.$$

17. Solution 1

    Notice that $\sqrt{3+2\sqrt{2}} = \sqrt{2}+1$ and $\sqrt{3-2\sqrt{2}} = \sqrt{2}-1$, so:

$$\sqrt{3+2\sqrt{2}} + \sqrt{3-2\sqrt{2}} = \mathbf{2\sqrt{2}}.$$

    Solution 2

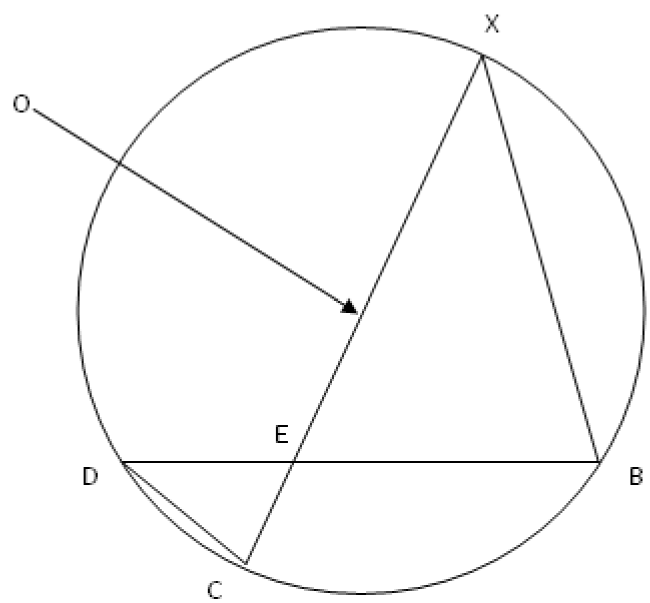
    Let $\sqrt{3+2\sqrt{2}} + \sqrt{3-2\sqrt{2}} = X > 0$, then:

$$\begin{aligned} X^2 &= (3+2\sqrt{2}) + (3-2\sqrt{2}) + 2\sqrt{3+2\sqrt{2}}\sqrt{3-2\sqrt{2}} \\ &= 6 + 2\sqrt{(3+2\sqrt{2})(3-2\sqrt{2})} \\ &= 6 + 2(3^2 - (2\sqrt{2})^2) = 6 + 2(9-8) = 8. \end{aligned}$$

Since $X > 0$, we have $X = \sqrt{8} = \mathbf{2\sqrt{2}}$.

Comment: Sometimes it is not clear what the individual square roots are, in which case the second method must be used. To apply solution 1 would require some trial and error; it becomes easier with experience.

18. Solution 1



Extend EO past O to meet the circle again at X. $\angle CDB = \angle CXB$ (angles subtended by a chord on the same side are equal), so now triangles $\triangle DEC$ and $\triangle XEB$ are similar. Hence:

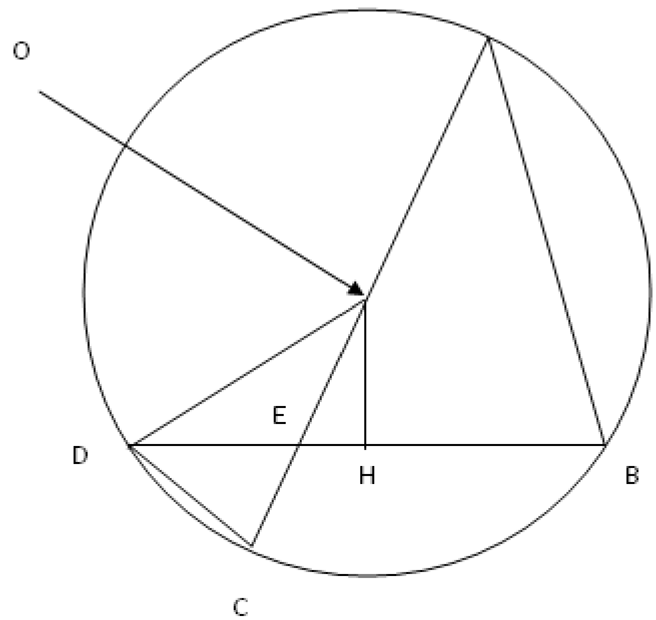
$$\frac{\overline{DE}}{\overline{EC}} = \frac{\overline{XE}}{\overline{EB}}$$

$$\overline{XE} = \frac{\overline{DE} \times \overline{EB}}{\overline{EC}} = \frac{3 \times 5}{1} = \mathbf{15}.$$

Now, the diameter $\overline{XC} = \overline{EC} + \overline{EX} = 1 + 15 = 16$, so the radius is $\mathbf{8}$.

Comment: The Power of a Point Theorem could have been applied; a more direct approach.

Solution 2



Consider the above diagram, let H be the foot of the altitude from O to DB (i.e. the point on BD so that $OH \perp BD$). Let R be the radius of the circle. Then,

$$\overline{DH} = \frac{1}{2} \times \overline{DB} = \frac{1}{2}(\overline{DH} + \overline{HB}) = \frac{1}{2} \times (3+5) = 4$$
$$\Rightarrow \overline{EH} = \overline{DH} - \overline{DE} = 4 - 3 = 1$$
$$\overline{OE} = \overline{CO} - \overline{CE} = R - 1$$

Pythagoras' Theorem on triangle $\triangle ODH$ yields:

$$\overline{OH}^2 + \overline{HD}^2 = \overline{OD}^2$$
$$\overline{OH}^2 = \overline{OD}^2 - \overline{HD}^2 = R^2 - 4^2 = R^2 - 16$$

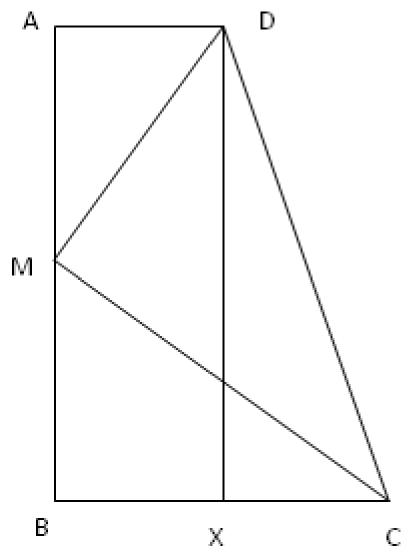
Pythagoras' Theorem on triangle $\triangle OED$ yields:

$$\overline{OH}^2 + \overline{EH}^2 = \overline{OE}^2$$
$$\overline{OH}^2 = \overline{OE}^2 - \overline{EH}^2 = (R-1)^2 - 1^2 = (R2 - 2R + 1) - 1 = R^2 - 2R$$

Combining the results gives:

$$\begin{aligned} R^2 - 16 &= R^2 - 2R \\ 2R &= 16 \\ R &= \mathbf{8}. \end{aligned}$$

19. Let $\overline{AM} = \overline{BM} = x$. Let X be the foot of the perpendicular from D to BC. Now $ABXD$ is a rectangle, since $\angle A = \angle B = \angle X = 90^o$.
$\therefore \overline{BX} = \overline{AD} = 4$ and $\overline{CX} = \overline{BC} - \overline{BX} = 8 - 4 = 4$.



Pythagoras' Theorem on triangles $\triangle ADM$, $\triangle BCM$, $\triangle MDC$ and $\triangle XDC$ respectively yield:

$$\begin{aligned}
\overline{DM}^2 &= \overline{AM}^2 + \overline{AD}^2 = x^2 + 16 \\
\overline{CM}^2 &= \overline{BM}^2 + \overline{BC}^2 = x^2 + 64 \\
\overline{CD}^2 &= \overline{DM}^2 + \overline{CM}^2 = x^2 + 16 + x^2 + 64 = 2x^2 + 80 \\
&= \overline{DX}^2 + \overline{CX}^2 = \overline{AB}^2 + \overline{CX}^2 = (2x)^2 + 4^2 = 4x^2 + 16.
\end{aligned}$$

From the last two lines, we see that:

$$2x^2 + 80 = 4x^2 + 16$$

$$\Rightarrow 2x^2 = 64$$

$$\Rightarrow \overline{CD}^2 = 64 + 80 = 144$$

$$\Rightarrow \overline{CD} = \mathbf{12}.$$

Comment: Timothy Loh has suggested a very elegant solution. Denote the midpoint of CD by K. Then the length $\overline{MK}$ will be 6 units. Notice that CD is the diameter of the circumcircle of $\triangle CMD$. Thus, $\overline{CK} = \overline{DK} = 6$ and $\overline{CD} = \mathbf{12}$.

20. Let their original ages be x and y. Now we can find positive integers, r and s, such that:

$$\begin{aligned}
r^2 &= 100x + y \\
s^2 &= 100(x+9) + y + 9 \\
&= 100x + y + 909 \\
&= r^2 + 909
\end{aligned}$$

$$\therefore s^2 - r^2 = 909 \Rightarrow (s-r)(s+r) = 1 \times 909 = 3 \times 303 = 9 \times 101.$$

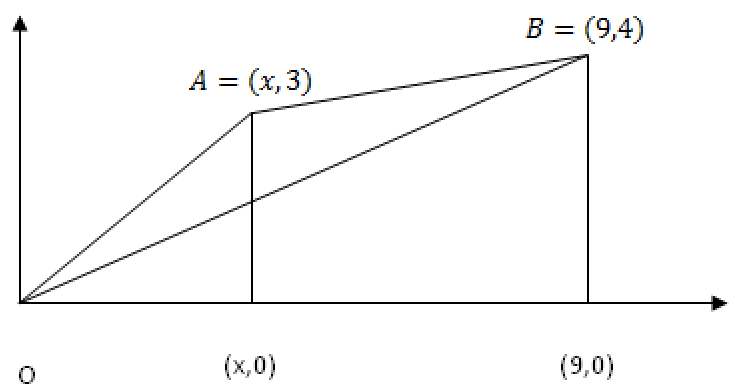
By assumption, $r, s > 0 \Rightarrow s + r > s - r$. Notice that 101 is prime, hence we obtain the following possibilities for $(s-r, s+r)$: $(1, 909)$, $(3, 303)$ and $(9, 101)$. Which give us the corresponding vaues of 455, 153 and 55 for s. Since s^2 is a four-digit number, we can disregard the first two cases. This leaves us with the final case. Now $100x + y = 46^2 = 2116$, so $x = 21$, $y = 16$ and

$$x + y = 21 + 16 = \mathbf{37}.$$

21. Let us think of the years as four-digit numbers, possibly beginning with 0. Then, there are eight possibilities beginning with 2: 2000, 2001, 2002, 2004,. . ., 2008. Otherwise such a number begins with 0 or 1. Each of these cases has the same number of possibilities. Now consider the other 3 digits:
Case 1: All three of them are 2. So there is one possibility: 222.
Case 2: Two of these 3 digits are 2. So there are 8 possibilities for the other digits, with 3 ways to place them. Hence there are $3 \times 8 = 24$ possibilities in total.
Case 3: One of these 3 digits is a 2. In which case, there are $8 \times 8 = 64$ possibilities for the other 2 digits, with 3 ways to place them. Hence there are $64 \times 3 = 192$ possibilities for this case.

Hence, the total number of such years is: $8 + 2(1 + 24 + 192) = \mathbf{442}$.

22. The diagram below gives us a geometric interpretation of this problem.



By Pythagoras' theorem, we have:

$$\begin{aligned} \overline{OA} &= \sqrt{x^2+9} \\ \overline{AB} &= \sqrt{y^2+1} \\ \overline{OB} &= \sqrt{9^2+4^2} = \sqrt{97} \end{aligned}$$

Now, $\sqrt{x^2+0} + \sqrt{y^2+1} = \overline{OA} + \overline{AB} \geq \overline{OB} = \sqrt{97}$, with the triangle inequality giving the last inequality.

Equality occurs when A lies on OB (specifically, when $x = \frac{9\times 3}{4} = 6.75$), so this must be when $\sqrt{x^2+9} + \sqrt{y^2+1}$ is minimised. Hence, the minimum value is $\sqrt{\mathbf{97}}$.

23. For any real number t, we can write $t = \lfloor t \rfloor + \{t\}$, where $\lfloor t \rfloor$ and $\{t\}$ are the integer and fractional parts of t, respectively. Strictly speaking, $\lfloor t \rfloor$ is defined as is the largest integer which is less than or equal to t, and $\{t\}$ is defined to be $t - \lfloor t \rfloor$. Let x and y, be the respective proportions of a full rotation, of the minute and hour hands of the clock measured clockwise from 12 O'Clock. E.G.: at 3 O'Clock, $x = 0$ and $y = 0.25$. Observe that $0 \leq x, y < 1$.

If (x, y) is legitimate, then $y = \{12x\}$. If (y, x) is legitimate, then $x = \{12y\}$. Now $12x - y = \lfloor 12x \rfloor$ and $12y - x = \lfloor 12y \rfloor$, so both of these are integers. Thus,

$$12(12x - y) + (12y - x) = 143x \in \mathbb{Z}.$$

So the only possibilities for x are $\frac{0}{143}, \frac{1}{143}, \ldots, \frac{142}{143}$. We claim that all of these give a legitimate solution (x, y), where $y = \{12x\}$.

Proof:
It is sufficient to show that $x = \{12\{12x\}\}$ for $x \in \{\frac{0}{143}, \frac{1}{143}, \ldots, \frac{142}{143}\}$. As LHS and RHS are both in the interval $[0, 1)$, it suffices to show that $x - \{12\{12x\}\}$ is an integer. However,

$$\begin{aligned} x - 12\{12x\} &= 144x - 12\{12x\} - 143x \\ &= 12(12x - \{12x\}) - 143x \\ &= 12 \times \lfloor 12x \rfloor - 143x \in \mathbb{Z}, \forall x \in \{\frac{0}{143}, \frac{1}{143}, \ldots, \frac{142}{143}\} \end{aligned}$$

Hence, there are **143** legitimate possibilities.

24.

$$\mathbf{6 \div (1 - \frac{3}{4}) = 24.}$$

25. We begin with the proof of a result that we will use later on.

McNugget Theorem: If m and n are relatively prime (i.e. they have no prime factors), and if doughnuts come in packets of m and n, then $mn - m - n$ is the largest number of doughnuts that cannot be bought. [This is a real theorem! The classic case deals with chicken nuggets rather than doughnuts.]

Proof:
Assume without loss of generality that $m < n$. Write the numbers $0, 1, 2, \ldots$ (yes, every non-negative integer!) in m columns. For example, if $m = 5$, write:

$$\begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 & 9 \\ & & \ldots & & \end{array}$$

Circle the first m multiples of n (the numbers $0, n, 2n, \ldots, (m-1)n$) in this chart. Because m, n have no shared prime factors, there will one circled number in each column. We observe that a number is a McNugget number if and only if it lies directly below a circled number.

Hence, the largest non-McNugget number lies directly above the largest circled number. The largest circled number is $m(n-1)$; the number directly above it is:

$$(m-1)n - m = mn - n - m.$$

Now that we've proved the theorem, let's use it. Since $6 \times 7 - 6 - 7 = 29$, anything of the form $29 + u (u \in \mathbb{Z}^+)$ is a McNugget number of $(6, 7)$, and 29 is not. Thus, anything of the form $145 + 5u (u \in \mathbb{Z}^+)$ is a McNugget number of $(30, 35)$, while 145 is not.

If we need to buy k doughnuts, our strategy will be to buy only enough 42-packs to make the remainder a multiple of 5. In the worst case scenario, we will need to buy four 42-packs. The largest number of doughnuts which cannot be bought will therefore be:

$$4 \times 42 + 145 = \mathbf{313}.$$

Comment: we didn't expect students to know the McNugget Theorem. However, such a result can be established through trial and error: try small pairs and find a pattern! In the above the solution, we somewhat arbitrarily chose 35 and 42 as our original pair. Choosing any other pair also works, and gives the same answer.

Finally, this was meant to be a very hard problem!