Schools Maths Olympics 2010
Solutions

1. Meat, pasta, cheese, pasta is repeated 1005 times, followed by one layer of meat. Hence **1006**.

2. It's **23**.

3. We see that $1 + 2 + 4 + 8 + 16 + 32 + 64 + 128 =$**255**.

4. $a + b = 43$ and $a + b + c = 66$, so $c =$**23**.

5. The angle next to it is $110° - 50° = 60°$, so $x = 180 - 75 - 60 =$**45**.

6. Let there be x ladies and y cats. Then $x + y = 22$ and $2x + 4y = 72$. Then $2y = 2x + 4y - 2(x + y) = 72 - 222 = 28$, so $y = 14$. Then $x = 22 - y = 8$. Hence **8 ladies, 14 cats**.

7. There are 5 students per team, 25 teams on average, 12 years, and students competed an average of 1.5 times, so the number of students who competed is $\frac{5\times 25\times 12}{1.5} =$ **1000**.

8. Basically try the small cubes, until **512**.

9. Julia is telling the truth, so Jiaying is, so Sam is, so Stephen is, so Yi is. Hence **1**.

10. Initially there are $\frac{84}{3} = 28$ women and $2\times 28 = 56$ men. Then 8 women join, making 36. The number of men must now be $\frac{4}{3} \times 36 = 48$, and so **8** men left.

11. There are $2 \times 5^3 =$ **250**.

12. Let the side lengths be x, y, z so that $xy = 12, xz = 25, yz = 27$. Multiplying the three equations yields

$$(xyz)^2 = 12 \times 25 \times 27 = 8100,$$

so the volume is $xyz =$ **90**.

13. He has a 50% chance of getting later-stage matches correct, and a $\frac{1}{2} \times \frac{3}{4} = \frac{3}{8}$ chance of getting a group match correct.

    Since he always has a 50% chance of picking either country, his success probabilities for each match are independent, so the overall probability of his feat is $(\frac{1}{2})^5 \times (\frac{3}{8})^3 = \frac{27}{2^{14}} = \mathbf{\frac{27}{16{,}384}}$.

14. Since triangles CDE and CEB share the same height (perpendicular to BC), the ratio of their heights is the ratio of their bases. Since triangles CEB and ABC share the same height (perpendicular to AB), the same can be said about them. Hence,

$$|CDE| = \frac{1}{3}|CBE| = \frac{1}{3} \times \frac{2}{3}|ABC| = \frac{2}{9}|ABC|.$$

Similarly,

$$|AEF| = |BFD| = \frac{2}{9}|ABC|.$$

Now

$$4 = |DEF| = |ABC| - |CDE| - |AEF| - |BFD| = \frac{1}{3}|ABC|.$$

$|ABC| = \mathbf{12}$.

15. We use Pythagoras' theorem. Initially, $PC = 17$ and $NP = \sqrt{17^2 - 8^2} = 15$. After Q is pulled down, $PC = 10$ and $NP = \sqrt{10^2 - 8^2} = 6$, so P has risen by $15 - 6 =$ **9 metres**.

16. If he goes into a shop with \$$x$, he comes out with \$$y =$ \$$(\frac{x}{2} - 1)$. Thus, $x = 2(y + 1)$. So Johnny goes into the 5th shop with \$2, the 4th shop with \$6, the third shop with \$14, the 2nd shop with \$30, and starts with **\$62**.

17. There are 7 equally likely observations: HHT, HTH, HTT, THH, THT, TTH, and TTT. Of these, two (HHT, THH) contain two heads in a row. Hence $\mathbf{\frac{2}{7}}$.

18. $\frac{n}{s(n)} = \frac{100x+10y+z}{x+y+z} = 100 - \frac{90y+99z}{x+y+z} \geq 100 - \frac{90y+99z}{1+y+z} = 100 - 90 - \frac{9z-90}{1+y+z} = 10 + 9 \times \frac{10-z}{1+y+z} \geq 10 + 9 \times \frac{10-z}{1+9+z} = 10 + 9 \times \frac{10-z}{10+z} \geq 10 + 9 \times \frac{10-9}{10+9} = 10 + \frac{9}{19} = \mathbf{\frac{199}{19}} = \frac{199}{s(199)}$.

19. It's **43**. It's easy to check that $44, \ldots, 49$ work, so everything higher than 43 works. Suppose $43 = 6a + 9b + 20c$ for integers $a, b, c \geq 0$. Since $43 \equiv 1 \pmod 3$ and $20 \equiv 2 \pmod 3$, we have $c \equiv 2 \pmod 3$, so $c \geq 2$. But now $6a + 9b \leq 3$, so $a = b = 0$, which doesn't work.

20. By Pythagoras' theorem,

$$AE^2 + FB^2 + DC^2 = (AP^2 - EP^2) + (BP^2 - FP^2) + (CP^2 - DP^2)$$

$$= (AP^2 - FP^2) + (BP^2 - DP^2) + (CP^2 - EP^2) = AF^2 + BD^2 + CE^2.$$

Now

$$AE = \sqrt{AF^2 + BD^2 + CE^2 - FB^2 - DC^2}$$
$$= \sqrt{12^2 + 8^2 + 13^2 - 6^2 - 14^2} = \sqrt{\mathbf{145}}.$$

21. The answer is 9376. It's not required, but we can prove that this is the only solution as follows: $n^2 \equiv n \pmod{10000}$, so

$$16|n(n-1)$$

$$625|n(n-1).$$

Then 625 divides n or $n-1$, and 16 divides the other.

If $625|n$ and $16|(n-1)$, we get $n = 625 + 10000k$ for some integer k, so there is no four-digit solution here.

Otherwise $16|n = 625m + 1$ for some integer m, so $16|(1+m)$. This leads to the only four digit solution: $m = 15$ yields $n = \mathbf{9376}$.

22. Without taking order into account, the possible sets of scores are 333330, 333321 and 333222. There are 6 possible orders of the first (six places to put the 0). There are 30 possible orders for the second (6 places to put the 2, then 5 for the 1, and $6 \times 5 = 30$). For the third, we need to choose three spots to put a 2:

123, 124, 125, 126, 134, 135, 136, 145, 146, 156,
234, 235, 236, 245, 246, 256,
345, 346, 356,
456

There are 20 possibilities here.

In total, $6 + 30 + 20 = \mathbf{56}$.

23. Let there be a articles per page and d advertisements per page, so that $d = 2(9-a)$.

Each day, $10000a^2$ Business Sections are sold, so the daily revenue is $10000a^2 \times 0.01d = 200a^2(9-a)$ per page.

The cost of the articles is $400a$ per page. As there are 10 pages, the daily profit is $2000a^2(9-a) - 4000a = 2000(9a^2 - a^3 - 2a)$.

Checking $a = 0, 1, 2, \ldots, 9$, we find that this is maximised when $a = 6$, in which case the profit is **\$192,000**.

24. Without loss of generality, let the cube have side length 1.

Each small tetrahedron has side length $\frac{1}{2}$. Its base has area

$$\frac{1}{2} \times \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2\sqrt{2}} = \frac{\sqrt{3}}{8}.$$

The height of each small tetrahedron is

$$\sqrt{(\frac{1}{\sqrt{2}})^2 - (\frac{1}{\sqrt{6}})^2} = \frac{1}{\sqrt{3}},$$

by Pythagoras' theorem.

The volume of a small tetrahedron is therefore

$$\frac{1}{3} \times \frac{\sqrt{3}}{8} \times \frac{1}{\sqrt{3}} = \frac{1}{24},$$

using the formula for the volume of a pyramid.

The volume of a large tetrahedron is therefore

$$2^3 \times \frac{1}{24} = \frac{1}{3},$$

using the fact that the volume ratio is the cube of the length ratio.

Let the intersection of the large tetrahedra have volume I. Then

$$8 \times \frac{1}{24} + 2I = 2 \times \frac{1}{3},$$

so $I = \frac{1}{6}$.

Finally, the volume of Han's cake is

$$2 \times \frac{1}{3} - I = \frac{2}{3} - \frac{1}{6} = \frac{1}{2}.$$

As the cube has area 1, **half** (**0.5** or $\mathbf{\frac{1}{2}}$) of it is 'wasted'.

25. Label the days Monday 1, Tuesday 2, ..., Sunday 7. Then there are 14 equally likely possibilities for one child: B1, B2, ..., B7, G1, G2, ..., G7.

In total there are 27 possibilities where at least one of the children is B2: $\{B2, B2\}, \{B2, other\}, \{other, B2\}$, so $1+2\times 13 = 27$ possibilities.

Thirteen of these are such that both are boys: $\{B2, B2\}, \{B2, B_{other}\}, \{B_{other}, B2\}$, and $1 + 26 = 13$.

The probability is therefore $\mathbf{\frac{13}{27}}$.