SMO 2011 solutions

1. After the eating of the legs there are 270 legs and 30 one-legged chickens so there are nine times as many legs as one-legged chickens. So in the pen there are \( \frac{54}{9} = 6 \) one-legged chickens.

2. 2 and 2

3. By Pythagoras’s theorem the zip line is 500 metres long. So Frederick will take \( \frac{500}{5} = 100 \) second on the zipline and \( \frac{700}{14} = 50 \) seconds on the ground. So he will take 150 seconds in total.

4. The last digits of the number \( 9^0, 9^1, 9^2, 9^3, \ldots \) are 1, 9, 1, 9, 1, \ldots hence the answer is 9. Alternatively \( 9^9 \equiv (-1)^9 \equiv -1 \equiv 9 \) modulo 10.

5. The sequence is 1, 1, 2, 4, 8, 16, 32, 64, 128, 256 (with nth term equal to \( 2^n - 2 \) for \( n > 1 \)). Hence 256.

6. \( x^3 - 7x + 6 = (x - 1)(x - 2)(x + 3) \) so the answer is \( 1^2 + 2^2 + (-3)^2 = 14 \)

7. Han eats \( \frac{1}{6} \) of the cake each minute and Jiaying eats \( \frac{1}{12} \) of the cake each minute. Together they eat \( \frac{1}{6} + \frac{1}{12} = \frac{1}{4} \) of the cake each minute. Hence the answer is 4 minutes.

8. The possibilities for the number of $1, $2 and $5 things are: (11,0,0), (9,1,0), (7,2,0), (5,3,0), (3,4,0), (1,5,0), (6,0,1), (4,1,1), (2,2,1), (0,3,1), (1,0,2). So the answer is 11.

9. 7

10. Let \( k, m, s \) be Kristian’s number, Mel’s number and Stephen’s number.
    Then \( s + \frac{1}{12} = (k + \frac{1}{m})(m + \frac{1}{k}) = mk + 2 + \frac{1}{mk} \). Clearly \( mk = 12 \) and hence \( s = 14 \).

11. Let \( H_c, S_c, Y_c \) be the number of pieces of chicken pizza that Han, Sam and Yi have respectively and let \( H_v, S_v, Y_v \) be the number of pieces of vegetarian pizza that Han, Sam and Yi have respectively. \( S_c = \frac{H_c + S_c + Y_c}{3} \) so \( S_c = \frac{H_c + Y_c}{2} \). Similarly \( S_v = \frac{H_v + Y_v}{2} \). Also \( H_c > S_c \) and \( H_v < S_v \). So \( 0 < H_v < S_v < Y_v = Y_c < S_c < H_c \). So the total number of pieces is \( H_v + S_v + Y_v + Y_c + S_c + H_c \geq 1 + 2 + 3 + 3 + 4 + 5 = 18 \). This can be achieved if \( H_v = 1, S_v = 2, Y_v = 3, Y_c = 3, S_c = 4 \) and \( H_c = 5 \). So the answer is 18.

12. \( \frac{m}{n} \leq \frac{n-1}{n} = 1 - \frac{1}{n} \) and none of 96,97,98,99 are the product of two primes. So \( n \leq 95 \). Therefore \( \frac{m}{n} \leq 1 - \frac{1}{n} \leq 1 - \frac{1}{95} = \frac{94}{95} \) which works. Hence the answer is \( \frac{94}{95} \).

13. \( AP = AC \cos 60 = \frac{12}{2} = 6 \) and \( AQ = AP \cos 60 = \frac{6}{2} = 3 \). So \( QC = AC - AQ = 12 - 3 = 9 \).
14. Let \( n \) be the number of ingredients in the dish. If the dish sells the net profit is \( 500 - 50n \) dollars. Therefore the expected profit is \( (500 - 50n)\frac{n}{10} = 5(10n - n^2) = 125 - 5(n - 5)^2 \) which is maximised when \( n = 5 \).

15. The sequence is 100, 50, 25, 77, 104, 52, 26, 13, 41, 56, 28, 14, 7, 23, 32, 16, 8, 4, 2, 1, 5 and then the last five numbers repeat. Since \( a_{21} = 5 \), \( a_{2011} = 5 \). So the answer is 5.

16. The clocks will next show the same time when they get 12 hours out of sync with each other. It takes the clocks one hour to move apart by 5 seconds. It takes the clocks 360 days = 8640 hours to move apart by 8640 minutes = 12 hours. Hence the answer is 360.

17. Let \( f(n) \) be the sum of the digits of \( n \). The for \( 0 \leq n < 1000 \), \( f(n) + f(999 - n) = 9 + 9 + 9 = 27 \). So \( 2 \sum_{i=1}^{999} f(n) = 2 \sum_{i=0}^{999} f(n) = \sum_{i=0}^{999} (f(n) + f(999 - n)) = \sum_{i=0}^{999} 27 = 27000 \). So the answer is \( \frac{27000}{2} = 13500 \).

18. Let \( b \) be the number of boys and let \( g \) be the number of girls. The number of pairs of a boy and girl who know each other is \( 4g = 5b \). The number of pairs of a dog and human who know each other is \( 3g + 2b = 46 \times 5 = 230 \). So \( 1150 = 15g + 10b = 23g \) so there are 50 girls at the party.

19. Let Heckyl’s age be the variable \( h \) and let Jeckyl’s age be the variable \( h + s \) (so \( s \) is constant). \( h|h + s \) if and only if \( h|s \). This will happen for the last time when \( h = s \). Let \( h_0 \) be Heckyl’s current age. Then \( h_0 + 8 = s \). Also \( h_0|s = h_0 + 8 \) so \( h_0|8 \) and nothing which is between \( h_0 \) and \( s \) divides \( s \). If \( h_0 = 1 \) then \( s = 9 \) but 3|9 and \( h_0 = 1 < 3 < 9 = s \). If \( h_0 = 2 \) then \( s = 10 \) but 5|10 and \( h_0 = 2 < 5 < 10 = s \). If \( h_0 = 4 \) then \( s = 12 \) but 6|12 and \( h_0 = 4 < 6 < 12 = s \). So \( h_0 = 8 \) so \( s = 16 \) so Jeckyl’s initial age is \( h_0 + s = 24 \).

20. Let the digits be \( a, b, c, d \). Then \( a + b + c + d = 10c + d \) so \( a + b = 9c \). Also 0 < \( a + b \leq 18 \) So \( a + b = 18 \) or \( a + b = 9 \). If \( a + b = 18 \) then \( a = b = 9 \) so \( 81|abcd = 99 \) a contradiction. So \( a + b = 9 \) and \( c = 1 \). Also \( abcd = 10a + b \). Therefore \( a|10a + b \) so \( a|a + b = 9 \) so \( a = 1, 3 \) or 9 and \( b|10a + b \) so \( b|10a + 10b = 90 \). If \( a = 1 \) then \( 8 = b|90 \) a contradiction. If \( a = 9 \) then \( 0 = b|90 \) a contradiction. So \( a = 3 \) and \( b = 6 \) so \( 36 = 3 \times 6 \times 1 \times d \) so \( d = 2 \). So \( n \) is 3612.

21. Let \( a \) be the expected time starting from the initial vertex. Let \( b \) be the expected time starting from a vertex which is adjacent to the initial vertex. Let \( c \) be the expected time starting from a vertex which is adjacent to the sugary vertex. Then \( a = b + 1, b = \frac{a + 2c}{3} + 1, c = \frac{2b}{3} + 1 \). So \( 3b = a + 2c + 3 = b + 2c + 4 \) so \( b = c + 2 \). So \( 2b + 3 = 3c = 3b - 6 \) so \( b = 9 \). Therefore \( a = 10 \).
22. If we look at the cube from a point on the line $AG$ we get the diagram above, a regular hexagon. So the angle between planes $CAYGE$ and $BAYGH$ is $60^\circ$. So the angle between faces $XAY$ and $BAY$ is $60^\circ$ and similarly the angle between faces $XBY$ and $BAY$ is $60^\circ$. Clearly faces $XAB$, $XAY$ and $XBY$ are mutually perpendicular. Also the angle between faces $ABY$ and $ABX$ is $45^\circ$. Therefore the sum of the angles between the faces is $60^\circ + 60^\circ + 90^\circ + 90^\circ + 90^\circ + 45^\circ = 435^\circ$

23. First place the emperor penguins. Paint one of the Emperor penguins pink. There are 12 ways of placing the pink one. Now there are 9 seats to put the other two in but they can’t sit next to each other. We can make a bijection between this and the ways of choosing any 2 of 8 seats by adding on empty seat between the two chosen from 8. So the number of ways of seating the last two emperor penguins is \( \binom{8}{2} \). So the number of ways of placing the emperor penguins, one of which is pink, is $12 \binom{8}{2}$. So the actual number is $4 \binom{8}{2}$ since they are actually identical. The number of ways of placing the macaroni penguins is now \( \binom{9}{3} \) and the number of ways of placing the fairy penguins is \( \binom{6}{3} \). So the number of ways of seating the penguins is $4 \binom{8}{2} \binom{9}{3} \binom{6}{3} = 4 \times 28 \times 84 \times 20 = 188160.$
24.

Label some of the vertices as shown. Using angle chasing all 6 of the triangles with labelled vertices are isosceles. Let $BC = 1$ and let $CD = x$. By the isosceles triangles $AB = BE = EC = CD = x$ and $ED = BD = x + 1$. Since $BEC$ is similar to $BDE$, 

\[ \frac{x + 1}{x} = \frac{BD}{BE} = \frac{BE}{BC} = x \text{ so } x^2 = x + 1 \text{ so } x = \phi. \] 

So $ED = \phi^2$. So the ratio in question is $\phi^4 = \phi^3 + \phi^2 = 2\phi^2 + \phi = 3\phi + 2 = \frac{7 + 3\sqrt{5}}{2}$

25.

It is fairly easy to see that the vertices of the prism will be on the surface of the sphere. Let the base of the prism be triangle $ABC$ with circumcentre $O_1$ and circumradius $R$. Let the centre of the sphere be $O$ and let the height of the prism be $h$. Using Pythagoras’s theorem $1 = OA^2 = OO_1^2 + O_1A^2 = \left(\frac{h}{2}\right)^2 + R^2$. So $4 = h^2 + 2R^2 + 2R^2 \geq 3\sqrt[3]{4h^2R^4}$ by the AM-GM inequality. So $\frac{4}{3\sqrt[3]{3}} \geq hR^2$. It isn’t too hard to prove that the area of $ABC$ will be maximised when $ABC$ is equilateral. In this case the area
of $O_1BC$ is $\frac{1}{2}O_1B \cdot O_1C \sin 120^\circ = \frac{\sqrt{3}}{4} R^2$. So the volume of the prism is $\frac{3\sqrt{3}}{4} R^2 h \leq 1$. Equality holds when $R = \frac{\sqrt{2}}{\sqrt{3}}$ and $h = \frac{2}{\sqrt{3}}$. So the answer is 1.