SMO 2011 solutions

1. After the eating of the legs there are 270 legs and 30 one-legged chickens so there are nine times as many legs as one-legged chickens. So in the pen there are $\frac{54}{9} = \mathbf{6}$ one-legged chickens.

2. **2 and 2**

3. By pythagoras's theorem the zip line is 500 metres long. So Frederick will take $\frac{500}{5} = 100$ second on the zipline and $\frac{700}{14} = 50$ seconds on the ground. So he will take **150** seconds in total.

4. The last digits of the number $9^0, 9^1 9^2, 9^3, \ldots$ are $1, 9, 1, 9, 1, \ldots$ hence the answer is **9**.

   Alternatively $9^9 \equiv (-1)^9 \equiv -1 \equiv 9$ modulo 10.

5. The sequence is 1,1,2,4,8,16,32,64,128,256 (with nth term equal to 2^{n-2} for $n > 1$.) Hence **256**.

6. $x^3 - 7x + 6 = (x-1)(x-2)(x+3)$ so the answer is $1^2 + 2^2 + (-3)^2 = \mathbf{14}$

7. Han eats $\frac{1}{6}$ of the cake each minute and Jiaying eats $\frac{1}{12}$ of the cake each minute. Together they eat $\frac{1}{6} + \frac{1}{12} = \frac{1}{4}$ of teh cake each minute. hence the answer is **4 minutes**.

8. The possibilities for the number of \$1, \$2 and \$5 things are: (11,0,0), (9,1,0), (7,2,0), (5,3,0), (3,4,0), (1,5,0), (6,0,1), (4,1,1), (2,2,1), (0,3,1), (1,0,2). So the answer is **11**.

9. **7**

10. Let k, m, s be Kristian's number, Mel's number and Stephen's number.

    Then $s + \frac{1}{12} = (k + \frac{1}{m})(m + \frac{1}{k}) = mk + 2 + \frac{1}{mk}$. Clearly $mk = 12$ and hence $s = \mathbf{14}$.

11. Let H_c, S_c, Y_c be the number of pieces of chicken pizza that Han, Sam and Yi have respectively and let H_v, S_v, Y_v be the number of pieces of vegetarian pizza that Han, Sam and Yi have respectively. $S_c = \frac{H_c+S_c+Y_c}{3}$ so $S_c = \frac{H_c+Y_c}{2}$. Similarly $S_v = \frac{H_v+Y_v}{2}$. Also $H_c > S_c$ and $H_v < S_v$. So $0 < H_v < S_v < Y_v = Y_c < S_c < H_c$. So the total number of pieces is $H_v + S_v + Y_v + Y_c + S_c + H_c \geq 1+2+3+3+4+5 = 18$. This can be achieved if $H_v = 1$, $S_v = 2$, $Y_v = 3$, $Y_c = 3$, $S_c = 4$ and $H_c = 5$. So the answer is **18**.

12. $\frac{m}{n} \leq \frac{n-1}{n} = 1 - \frac{1}{n}$ and none of 96,97,98,99 are the product of two primes. So $n \leq 95$. Therefore $\frac{m}{n} \leq 1 - \frac{1}{n} \leq 1 - \frac{1}{95} = \frac{94}{95}$ which works. Hence the answer is $\boldsymbol{\frac{94}{95}}$.

13. $AP = AC\cos 60 = \frac{12}{2} = 6$ and $AQ = AP\cos 60 = \frac{6}{2} = 3$. So $QC = AC - AQ = 12 - 3 = \mathbf{9}$.

14. Let n be the number of ingredients in the dish. If the dish sells the net profit is $500 - 50n$ dollars. Therefore the expected profit is $(500-50n)\frac{n}{10} = 5(10n - n^2) = 125 - 5(n-5)^2$ which is maximised when n=**5**.

15. The sequence is 100, 50, 25, 77, 104, 52, 26, 13, 41, 56, 28, 14, 7, 23, 32, 16, 8, 4, 2, 1, 5 and then the last five numbers repeat. Since $a_{21} = 5$, $a_{2011} = 5$. So the answer is **5**.

16. The clocks will next show the same time when they get 12 hours out of sync with each other. It takes the clocks one hour to move apart by 5 seconds. It takes the clocks 360 days = 8640 hours to move apart by $8640 \times 5 = 43200$ seconds =720 minutes =12 hours. Hence the answer is **360**.

17. Let $f(n)$ be the sum of the digits of n. The for $0 \leq n < 1000$, $f(n) + f(999-n) = 9+9+9 = 27$. So $2\sum_{i=1}^{999} f(n) = 2\sum_{i=0}^{999} f(n) = \sum_{i=0}^{999}(f(n)+f(999-n)) = \sum_{i=0}^{999} 27 = 27000$. So the answer is $\frac{27000}{2} =$ **13500**.

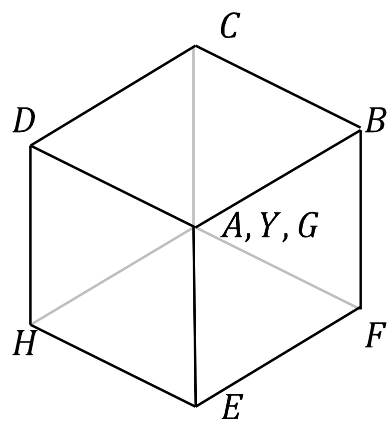
18. Let b be the number of boys and let g be the number of girls. The number of pairs of a boy and girl who know each other is $4g = 5b$. The number of pairs of a dog and human who know each other is $3g + 2b = 46 \times 5 = 230$. So $1150 = 15g + 10b = 23g$ so there are **50** girls at the party.

19. Let Heckyl's age be the variable h and let Jeckyl's age be the variable $h+s$ (so s is constant). $h|h+s$ if and only if $h|s$. This will happen for the last time when $h = s$. Let h_0 be Heckyl's current age. Then $h_0 + 8 = s$. Also $h_0|s = h_0 + 8$ so $h_0|8$ and nothing which is between h_0 and s divides s. If $h_0 = 1$ then $s = 9$ but $3|9$ and $h_0 = 1 < 3 < 9 = s$. If $h_0 = 2$ then $s = 10$ but $5|10$ and $h_0 = 2 < 5 < 10 = s$. If $h_0 = 4$ then $s = 12$ but $6|12$ and $h_0 = 4 < 6 < 12 = s$. So $h_0 = 8$ so $s = 16$ so Jeckyl's initial age is $h_0 + s =$ **24**.

20. Let the digits be a, b, c, d. Then $a + b + c + d = 10c + d$ so $a + b = 9c$. Also $0 < a + b \leq 18$ So $a + b = 18$ or $a + b = 9$. If $a + b = 18$ then $a = b = 9$ so $81|abcd = 99$ a contradiction. So $a+b = 9$ and $c = 1$. Also $abcd = 10a+b$. Therefore $a|10a + b$ so $a|a + b = 9$ so a =1,3 or 9 and $b|10a + b$ so $b|10a + 10b = 90$. If $a = 1$ then $8 = b|90$ a contradiction. If $a = 9$ then $0 = b|90$ a contradiction. So $a = 3$ and $b = 6$ so $36 = 3 \times 6 \times 1 \times d$ so $d = 2$. So n is **3612**.

21. Let a be the expected time starting from the initial vertex. Let b be the expected time starting from a vertex which is adjacent to the initial vertex. Let c be the expected time starting from a vertex which is adjacent to the sugary vertex. Then $a = b + 1$, $b = \frac{a+2c}{3} + 1$, $c = \frac{2b}{3} + 1$. So $3b = a + 2c + 3 = b + 2c + 4$ so $b = c + 2$. So $2b + 3 = 3c = 3b - 6$ so $b = 9$. Therefore $a =$ **10**.

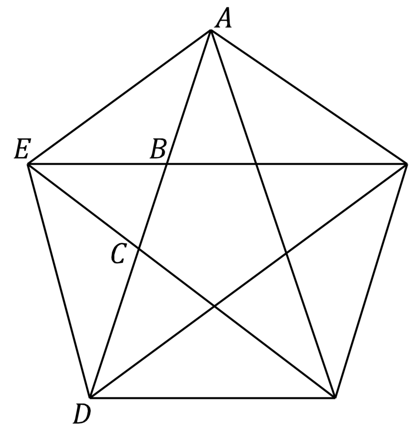
22.



If we look at the cube from a point on the line AG we get the diagram above, a regular hexagon. So the angle between planes $CAYGE$ and $BAYGH$ is $60°$. So the angle between faces XAY and BAY is $60°$ and similarly the angle between faces XBY and BAY is $60°$. Clearly faces XAB, XAY and XBY are mutually perpendicular. Also the angle between faces ABY and ABX is $45°$. Therefore the sum of the angles between the faces is $60° + 60° + 90° + 90° + 90° + 45° = \mathbf{435}°$

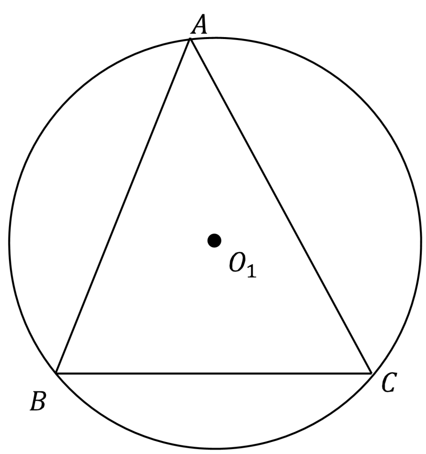
23. First place the emperor penguins. Paint one of the Emperor penguins pink. There are 12 ways of placing the pink one. Now there are 9 seats to put the other two in but they can't sit next to each other. We can make a bijection between this and the ways of choosing any 2 of 8 seats by adding on empty seat between the two chosen from 8. So the number of ways of seating the last two emperor penguins is $\binom{8}{2}$. So the number of ways of placing the emperor penguins, one of which is pink, is $12\binom{8}{2}$. So the actual number is $4\binom{8}{2}$ since they are actually identical. The number of ways of placing the macaroni penguins is now $\binom{9}{3}$ and the number of ways of placing the fairy penguins is $\binom{6}{3}$. So the number of ways of seating the penguins is $4\binom{8}{2}\binom{9}{3}\binom{6}{3} = 4 \times 28 \times 84 \times 20 = \mathbf{188160}$.

24.



Label some of the vertices as shown. Using angle chasing all 6 of the triangles with labelled vertices are isosceles. Let $BC = 1$ and let $CD = x$. By the isosceles triangles $AB = BE = EC = CD = x$ and $ED = BD = x+1$. Since BEC is similar to BDE, $\frac{x+1}{x} = \frac{BD}{BE} = \frac{BE}{BC} = x$ so $x^2 = x + 1$ so $x = \phi$. So $ED = \phi^2$. So the ratio in question is $\phi^4 = \phi^3 + \phi^2 = 2\phi^2 + \phi = 3\phi + 2 = \mathbf{\frac{7+3\sqrt{5}}{2}}$

25.



It is fairly easy to see that the vertices of the prism will be on the surface of the sphere. Let the base of the prism be triangle ABC with circumcentre O_1 and circumradius R. Let the centre of the sphere be O and let the height of the prism be h. Using Pythagoras's theorem $1 = OA^2 = OO_1^2 + O_1A^2 = \left(\frac{h}{2}\right)^2 + R^2$. So $4 = h^2 + 2R^2 + 2R^2 \geq 3\sqrt[3]{4h^2R^4}$ by the AM-GM inequality. So $\frac{4}{3\sqrt{3}} \geq hR^2$. It isn't too hard to prove that the area of ABC will be maximised when ABC is equilateral. In this case the area of O_1BC is $\frac{1}{2}O_1B \cdot O_1C \sin 120^\circ = \frac{\sqrt{3}}{4}R^2$. So the volume of the prism is $\frac{3\sqrt{3}}{4}R^2h \leq 1$. Equality holds when $R = \frac{\sqrt{2}}{\sqrt{3}}$ and $h = \frac{2}{\sqrt{3}}$. So the answer is **1**.