MUMS

PRESIDENT: Tien Hyung
tlhuynh at student.unimelb.edu.au

VICE-PRESIDENT: Damian Pavlyshyn
damian.pavlyshyn at gmail.com

SECRETARY: Yuhe Gong
ronziegong at gmail.com

TREASURER: Aaron Chong
awychong at hotmail.com

EDITOR OF Paradox: Ben Hague
paradox.editor at gmail.com

EDUCATION OFFICER: Paul Nguyen
paulpn14 at gmail.com

PUBLICITY OFFICER: Michael Rixon
mrixon at student.unimelb.edu.au

POST-GRAD REP: Muhummad Adib Surani
adib.surani at gmail.com

UNDER-GRAD REP: Ben Hines
bhines at student.unimelb.edu.au

UNDER-GRAD REP: James F Boon
jboon at student.unimelb.edu.au

UNDER-GRAD REP: Britt Morison
brittmorison at gmail.com

UNDER-GRAD REP: Will Troiani
william.a.troiani at gmail.com

UNDER-GRAD REP: Daniel Johnston
djlikesdjs at gmail.com

UNDER-GRAD REP: Allan Pennings
penningallan at gmail.com

MUMS WEBSITE: www.mums.org.au
MUMS EMAIL: mums@ms.unimelb.edu.au
MUMS PHONE: (03) 8344 4021
MUMS TWITTER: http://twitter.com/MelbUniMaths
In This Edition of Paradox

Interview: John Sader
*Ruwan Devasurendra*

Puzzle Page
*Jason Tang and Sarah Boland*

Biography: Emmy Noether
*Ruwan Devasurendra*

Knot 4
*Dougal Davis*

---

**WEB PAGE:** www.ms.unimelb.edu.au/~mums/paradox  
**E-MAIL:** paradox.editor@gmail.com  
**PRINTED:** Wednesday, 27 May 2015  
**COVER:** Emmy Noether, the subject of this issue’s biography by Ruwan Devasurendra

---

**Paradox wants you!**

**Want to share your interest in maths and stats with others?**
Paradox is always on the look-out for contributors, especially given the semester break is approaching. Whether it is an article on an area or application of maths and stats that interests you, an interesting maths-related film or book you saw, or a summary of your research, Paradox would love to hear from you.
Paradox is also recruiting editors to join our editorial team, to help proof-read and typeset articles. No experience required – just a keen eye for detail and enthusiasm. Experience in LaTeX desirable for typesetting roles but some brief training could be provided.

**Email:** paradox.editor@gmail.com for more information or to express your interest.
The Editor’s Column

Well what a semester it has been! The AGM was held in mid-April with a turnout of approximately 40 committed members who graciously took it upon themselves to elect themselves to positions on our esteemed committee. A full run-down of who’s who on the new committee will appear in Issue 3, 2015 at the beginning of Semester Two. Evidently I am still here, and, as will become evident if you turn to the next page, we have a new President, Tien Hyung, who continues the tradition of populating a page of this publication with his wise words. For now, you will have to satiate your desire to get to know the new committee by looking at the names and email addresses on the inside cover and keenly await the next issue.

While speaking of issues – an issue that confronted some MUMSians I spoke to was whether MUMS and PPS were merging, as was announced in a special issue of Paradox that arrived in hard-copy only, suspiciously close to April Fools’ Day\(^1\). I would like to thank my esteemed colleague Shanette De La Motte for being unable to decipher my writing on the MUMS whiteboard and asking Past President Mel Chen if there was a ‘Parody’ happening! Just got more and more silly from there and the end result was 8 pages of complete nonsense\(^2\).

But why should I be talking about past ‘issues’ when there is a current issue comprised of these glorious pages that you hold in your hands\(^3\)? We finally have the rousing conclusion of Dougal Davis’ epic Knot series\(^4\). We also have new puzzles, set out by first-time contributors Jason Tang and Sarah Boland.

I would also like to express my thanks to regular contributor Ruwan Devasurendra who for this issue has provided an interview with Prof. John Sader and a biography on mathematician Emmy Noether.

Best wishes for exams and the winter break!

— Ben Hague

\(^1\)If any one missed out come to the MUMS room and find it in the large Australia Post box of past Paradox Issues.
\(^2\)But I never said it wasn’t enjoyable!
\(^3\)Or for our online readers, the collection of glorious bytes that have been downloaded to your computer and beamed onto your screen
\(^4\)He acknowledges that whilst it isn’t actually ‘finished’ there will be no more puzzles, however a solution may be provided in the future
Words from the President

Hello MUMS, on behalf of the new Committee, I would like to say thank you for all of those who came and voted for us at the Annual General Meeting on the 16th April 2015. We’re eager and ready to serve the Maths community!

One of MUMS biggest events of the year, the annual Puzzle Hunt has been successfully organized. Puzzle Hunt 2015 has been jointly won by teams A New Plugh and The Judean GNUS, who found the X-wing fighter on Saturday 9th May at approximately 2:50 pm. Awesome job! A big thanks is given to our lovely graduate community for helping us organize the event!

Looking forwards to the end of the year, the MUMS Committee are excited to serve the community through organizing: the Annual Schools Maths Olympics on the 16th of August, the Trivia Night on the 28th of May, University Maths Olympics and Games Night.

Hope that everyone will have a productive SWOTVAC period and smash their Maths Exams!

When University starts again next semester, come and join us at one of our weekly Seminars!

Stay tuned!

— Tien Hyung

Trivia Night

Fancy some fun on a Friday evening? Come along to MUMS’ bi-annual trivia night. Snacks, drinks and questions provided. Bring a team of 4 or 5 or form one on the night. 515pm for a 530pm start, Friday 29th May, Russell Love Theatre, Richard Berry
Interview with Prof John Sader: Multi-project Mathematician

John Sader is a Professor at the University of Melbourne who, among teaching several subjects, spends his research endeavours on a range of different projects. He generously set aside some time to talk to me about his current work and his journey thus far.

How did you first get involved in mathematics?

So, my undergraduate and Ph.D. are not in mathematics, but electrical engineering, and I completed them at UNSW. The engineering side of things was enjoyable, however it was the maths that really interested me the most. Back then, only two of us in our cohort went on to do a Ph.D. – it was not that common for people to be doing a Ph.D., especially in engineering! I remember some of the lecturers advising us, “Why do you want to do a Ph.D.?”. If you answered with the intent of getting a job, they would suggest you do a Masters, but if you answered with the desire to do research, they would suggest the Ph.D.

Given my interests, I did a very mathematical project for my Honours project, and I enjoyed that so much I went on to a Ph.D. I did a little bit of lab work, but I didn’t really like that and quickly realised that it was the analysis of the project – solving Maxwell’s equations for light – that really had me interested.

So was this analysis of Maxwell’s equations for light for your Honours or Ph.D.?
Both! The department placed a whole list of Honours projects in front of us, and they spanned topics such as power engineering, systems engineering, and so on, but what really caught my attention was the field of optical communication, which was relatively new at the time, and I ended up working on a project for that. However, I ended up switching fields completely when I came here. My background was all in electrical engineering, but I had the opportunity to work with Professor Lee White when I came to Melbourne – do you know him?

I have heard his name mentioned within the Department before. Could you tell us a bit more about him?

He is a very smart man, and I think he just retired. He used to be in this office! I remember when I first came here for the position, I walked in here and he said to me, “Let’s go to Lygon St – this will be the first and last time I buy you a coffee!” We have of course bought coffee together since then, but he is a very, very good applied mathematician. He is very intuitive. While some people focus on learning mathematical techniques, Lee had both technique and a natural feel for the way things worked. This enabled him to solve problems others couldn’t.

One time there was a guy from industry in here in the office, and presented a problem in pharmaceutics that involved moving fluids around. The guy said that he was trying to design a device for a particular process, and he had come for advice. Over a period of about two hours, Lee started writing on his board, formulating the problem. He would ask for advice, but he was flying along! At the end of it, he boiled it down to a little equation, and hands it to the guy: “Here you go.” I’ve met some very, very sharp people, but most people would rather take a night to work things out in private, and never do this kind of thing live, so it was very impressive.

How did you find Prof. White’s tutelage?

Lee led by example, and was very inspirational by virtue of his natural talents. He could see the solution to a problem from the beginning. As an applied mathematician, that is what you aim for: to be able to see the solution before you have solved the problem. You need to have a feel for where things are going. Often it is your intuition or physical understanding that guides the maths, rather than the maths guiding your intuition.

With that, Lee drove you to push the boundaries, and encouraged you to do
what you wanted to do to the best of your ability.

**What other fields have you had a chance to work in?**

So when I came here, I went from optics into solid mechanics. Lee gave me a bunch of different projects, and then he told about this new instrument called the atomic force microscope. He told me that they needed to measure spring constants for these tiny things they call cantilevers. He asked me whether I had worked in the field before, and when I said no, he handed me a copy of Landau & Lifshitz, one of the bibles of physics...

[At this point, John gets up, collects the copy from his shelf, and opens the book to its contents page]

Look at all these topics here! This book is very deep and very good for research, but it is so high level that it is difficult to learn from it! So I went down to the engineering library and pulled out some other books on solid mechanics, and eventually solved the problem that he wanted and we submitted our first paper. The field was pretty new back then and people really liked it, so it took off from there. Then I started moving fields and looked at things like fluid mechanics, and even today I’m doing different problems. Have a look here: [shows me a small stack of library books on his desk]. Gas dynamics, microelectromechanical systems, aerodynamics and high speed wing theory – so you can see here how some of these things we are dealing with are on a very small scale, whereas others are very large. We’ve even worked on spacecraft technology.

I like moving around different fields and learning something new, and applying what I’ve learnt in one area to another area. Working on the same problem for your whole career... that doesn’t interest me whatsoever. I’ll often jump and start working in different areas, just so I can learn something new and start thinking about completely different things. I do that very often.

Doing Honours, or your Ph.D., you have to focus and understand the problem very, very well. But once you have finished your Ph.D., branch out, and explore new areas. People often continue on in the area of their Ph.D., which is fine, but I didn’t, and that made it hard actually. If you jump fields, whilst others are running in their original field, you’re essentially restarting again. It opens your eyes to different things, and that drives you to work harder. In the long run, this enables you to tackle a wide range of problems, which is very useful as an applied mathematician. This allows you to also bring a new per-
spective to a problem, since you haven’t been trained in that particular field’s school of thought during your undergrad and Ph.D., and it often allows you to create fresh and novel solutions that may seem simple on your end, but be groundbreaking for those in the field.

**One such contribution from yourself is the ‘Sader method’ of calibration. Could you tell us how it works?**

Um...are you familiar with how the atomic force microscope works?

**Um...no.**

Have a look at this then: [He collects a small box off his shelf, that contains a shiny, pointed piece of silicon less than a centimetre-square in area, sitting atop a felt-like surface; he points to the tip of the silicon piece]. Are you short-sighted?

**Yes.**

I am too...it helps. Take your glasses off and bring this close to your eyes. Look very carefully. You should be able to see a little needle sticking out of one of the sides of the tip. That thing there is called a cantilever. You can think of it as a little ruler, with a tip at the end of it that is extremely sharp – it can get to a 5nm radius of curvature at the end.

So what they do is they run this ruler with the needle tip over a surface. As the needle moves over the surface, the ruler goes up, and it goes down. Before you do that, you shine a laser on the back of it and it is reflected towards a split photodiode. Hence, as the ruler goes up and down, the laser spot moves as well, and hence the photodiode can detect the deflection. The remarkable thing is that you can now measure displacements at the atomic scale, so fractions of a nanometre. So when this thing goes over an atom, you can see it! And that’s why it’s called an atomic force microscope: it uses force to measure surfaces with atomic resolution.

So, one thing that users of the instrument want to know about the cantilever is its stiffness. You would have done mechanics, as you’re in Biomedical Engineering, right? Have a look at this: [begins writing]

\[ k = m\omega^2 \]

The radial frequency, \( \omega \), of the cantilever is related to its stiffness and its mass. So the stiffer it is, the higher the frequency; the heavier it is, the lower the
frequency. So the idea here is that if you measure this thing vibrating – which you can, since it’s so small that with the detection scheme you can measure its Brownian motion – and if you know what the mass is, you can work out its stiffness. But what is the mass of the cantilever? We don’t know, and the masses of cantilevers can vary a lot, since they can put coatings on them too.

So what I did many years ago was that I solved the problem of what happens when you take the cantilever beam and vibrate it in fluid. Then what I did was take this fluid theory and apply it to the equation from before. I replace the cantilever mass m, and \( \omega \), the frequency in vacuum, with the frequency in fluid and also the Q in fluid – remember air is a fluid! Now Q is measure of the damping; it’s like an inverse damping constant. So the higher the Q, the lower the damping. So if you measure displacement as a function of frequency, and take the Fourier transform, you will get a \( \delta \)-function if you have the one frequency. But there is always damping present and it spreads out that frequency, so you end up with a broad peak.

And you measure this. If you take the vibrating cantilever, take the signal out, and take the Fourier transform of this timeseries and analyse it in a special way, you get out the \( \omega \) and Q for the fluid – air is most commonly used of course. With these in hand, you replace the frequency in a vacuum and the mass of the beam with these two values, and you’re done\(^1\).

This became very popular because people can simply look at the cantilever now, and measure its stiffness. The instrument can also measure tiny forces: if

\(^1\)Prof. Sader has since created a free app that computes the stiffness as a function of the plan dimensions of the cantilever, the desired frequency, and the desired Q.
you know the stiffness of its cantilever, and it deflects a certain amount, you can use Hooke’s Law to extract forces. This allows you to bring two atoms close to each other and measure the force between them, or you can go and measure friction between an atom and a surface. People have actually done this! We published the method first in 1999, and have developed it further since then. It used to apply for straight beams, but we have now generalised it any shape of beam. Companies now use this in their instruments. It was a really nice application of mathematics, and the key to it was understanding what the end users, the clients, wanted: their problem concerned one parameter that they could not measure, so we replaced it with two that they could.

**What advice would you give to students who may want to begin a career like yours?**

Let me give some more general advice: don’t do what someone wants you to do, do what you want to do. I mean, I was rather fortunate to find something that I enjoy doing. When I first finished my Ph.D., I had no desire to teach, I just wanted to do research, but now I really enjoy teaching, and the students seem to like me. I would encourage students to take good opportunities when they come up. Things may come up that are completely unexpected which may be good for you: you won’t know until you try them!

I had a student in here last week who was asking me whether he should do actuarial studies, and I asked him why he was interested, and he replied that they made a lot of money. I told straight out not to chase the money! It’s not going to make you happy. I know of many people that chase the money and aren’t happy. I know of one guy who worked at a bank in Sydney: he used to have a massive pay packet. You know what he is doing now? He is retraining to be a teacher! Teaching is a great profession, but I mean he knows that he will be getting a massive cut in salary, so why is he doing this? Because he wants to teach, and he’s going to be much happier doing that than he is making a hell of a lot of money in the bank.

Ultimately, try and find something that you enjoy doing, otherwise you’ll be always look forward to the weekend and be depressed by the time Monday comes around. If you find something that you like doing, follow it!

— Ruwan Devasurendra
MUMS Puzzles

Editor’s Note: Solutions are provided on page 16 of this issue

1. An 8x8 chessboard can be tiled by 2x1 dominoes, with none overlapping, in 258584046368 different ways. If I cut off the bottom-left and top-right squares, so that 62 squares remain, how many ways are there to tile the remaining mutilated chessboard?

2. Blue checkers can only move right and red checkers can only move left. The goal is to get the three blue checkers onto the right and the three red checkers onto the left, using the following rules: i) If the space directly in front of the checker is empty, it may move there. ii) If the space directly in front of the checker is occupied by an opposite-coloured checker and the square on the other side is empty, the checker may jump over the intermediate checker and land on the other side. What is a sequence of legal moves which will accomplish the goal?

3. There are six caves in a row numbered 1-6 such that 1 is only adjacent to 2, 2 is adjacent to 1 and 3, etc. There is a hermit who resides in a cave, but he always moves to an adjacent cave at the end of every day. Your goal is to find the hermit, but you are allowed to search only one cave per day. Can you formulate a sequence of searches so that you can be sure of finding the hermit, regardless of his initial cave and subsequent movements?

— Jason Tang

4. During a recent census, a person told the census taker that ze\(^1\) had three children, all named Gertrude. When asked their ages, ze replied, “The product of their ages is 72. The sum of their ages is the same as my house number.”

The census taker ran to the door and looked at the house number. “I still can’t tell,” the census taker complained. The resident replied, “Oh that’s right, I forgot to tell you that the oldest one likes chocolate pudding.”

\(^1\)in the interest of gender neutrality the term for someone who prefers to not classify their gender has been used
The census taker promptly wrote down the ages of the 3 children. How old are they?

— Sarah Boland

Emmy Noether (1882–1935)

“\textit{In the judgment of the most competent living mathematicians, Fräulein Noether was the most significant creative mathematical genius thus far produced since the higher education of women began.}”\textsuperscript{1}

— Albert Einstein, writing to The New York Times on Noether’s death

Riemann was a German mathematician who made significant contributions to many areas, including analysis, geometry, physics and philosophy. The boost given to pure and abstract mathematics in 19th century Germany saw Riemann cross paths with many other influential mathematicians in his lifetime.

Early life, education and research

Noether was born in the Bavarian city of Erlangen, and was the eldest of the four children of Max Noether, a mathematician, and Ida Amalia Kaufmann, a wealthy merchant’s daughter. Though Noether “did not appear to be exceptional as a child”\textsuperscript{2}, she was a clever and sociable young girl. As with many


girls of the age, she was taught domestic duties and was given piano lessons – none of which she especially enjoyed, save for her passion for dance\(^3\).

Notably, Noether was proficient enough in French and English that she could have gone on to teach these languages at girls’ schools, however she instead chose to enrol in the University of Erlangen to study mathematics. This posed several problems, as the academic staff and senate had deemed it unfit for mixed-sex classes to commence, and instead only allowed Noether to audit classes, as opposed to being a regular student, and even then only with the individual consent of each lecturer whom she wanted to learn from.

Despite this discrimination, she graduated in 1903, and returned in 1904 when gender restrictions on mathematical study were relaxed. Noether’s peers included Hermann Minkowski, Felix Klein and David Hilbert, the lattermost whom would become a staunch advocate for Noether’s rights. Her Ph.D. dissertation of 1907 was on algebraic invariants and was entitled, On Complete Systems of Invariants for Ternary Biquadratic Forms. Therein, she continued her own research and assisted her father at Erlangen, all without pay.

In 1915, Klein and Hilbert invited Noether to come to the University of Göttingen\(^4\) to help use her knowledge of algebraic invariants on exploring Albert Einstein’s recently published theory of general relativity\(^5\). However, the University’s philologists and historians objected to a woman joining the academic ranks, and prevented her from being fully recruited. Hilbert was incensed, and it is here that he made his famous remark:

“I do not see that the sex of the candidate is an argument against her admission as privatdozent. After all, we are a university, not a bath house.”\(^6\)

To circumvent this, it was arranged that Noether would deliver her lectures under Hilbert. Once more, her work for the University was unpaid and without official title.

---

\(^3\)Osen, L.M. Emmy (Amalie) Noether (MIT Press, 1974), 141 - 152.
\(^4\)Göttingen was one of the most talent-rich mathematical hubs of the late 19th/early 20th century, and boasted alumni such as Gauss, Riemann and Dirichlet.

\(^6\)Thompson, W.J. Angular Momentum: an illustrated guide to rotational symmetries for physical systems, (Wiley, 1994), 5.
Noether’s Theorem and abstract algebra

Proven by Noether in 1915 and published in 1918, the theorem states:

*If a system has a continuous symmetry property, then there are corresponding quantities whose values are conserved in time.*

Mechanically, this means that the Lagrangian of a symmetric system will be preserved regardless of translations in space and time; or, in a system with symmetry in space and time, the Lagrangian is space and time invariant, which results in conservations of linear momentum and energy respectively.

Noether’s most significant work in the field of abstract algebra began in 1920, with a paper that defined left and right ideals in a ring. A year later, her paper *Idealtheorie in Ringbereichen* analysed ascending ring conditions, and the descriptive term “Noetherian” with respect to mathematical objects has arisen since.

Later life, death and legacy

The rise of Nazi Germany saw Noether, a Jew, being expelled from Göttingen in 1933. Even though she had had her teaching rights revoked, Noether did not see this as a hurdle – perhaps since her teaching rights were never fully recognised to begin with – and continued her research independently. She is noted as providing comfort and support to those who had similarly been forced to leave, before she managed to secure a position at Bryn Mawr College in the United States. Upon invitation in 1934, she also delivered a series of lectures at Princeton University, but remarked that there she was not welcome, at a “men’s university, where nothing female is admitted”.

In 1935, Noether received surgery for a large ovarian cyst and appeared to be recovering well, however she was struck with a virulent infection and passed away as consequence.

Thereafter, the tributes to Noether from her colleagues began to circulate, including Einstein’s famous letter to *The New York Times*. Though Noether was never appointed a full professor, her work in algebra and theoretical physics lay the groundwork for many significant discoveries by her successors. Fur-

---

thermore, Noether is credited as being especially generous with her ideas, and as such many of her students and colleagues have had the benefit of her insight and creativity to spur on, guide and inspire further research.

— Ruwan Devasurendra

**MUMS Puzzles Solutions**

1. 0 - each domino covers one black square and one white square, but my mutilated chessboard has 30 black squares and 32 white squares.

2. B1, R1, R2, B1, B2, B3, R1, R2, R3, B1, B2, B3, R2, R3, B3

3. The search sequence 23455432 is the quickest way to be certain of finding the hermit.

4. The children are aged 3, 3 and 8. From the problem you know that if you denote the children’s ages as a, b and c then you have the following conditions:

   (1) \( a + b + c = \) house number

   (2) \( abc = 72 \)

   (3) \( a \leq b < c \) (as you can’t have twins being the eldest)

(4) additionally you also know the house number must be such that there is only one set of values for a, b and c that does not violate condition (3).

The solution can be found by working up through possible house numbers starting from 1 (the first 10 won’t take you very long!), then trying different values for a and solving for b and c.

---

Student: “My math teacher is crazy”.
Parent: “Why?”
Student: “Yesterday she told us that five is 4+1; today she is telling us that five is 3 + 2.”
Knot 4 (with apologies to Lewis Carroll)

“Well now”, said the dragon to his prisoners, “our new friends have given me a wonderful idea for a puzzle! I’m going to ask one of my slaves in the village to bring me sixteen hats tomorrow morning, each a different colour. Once we have the hats, you will all line up in single file, and I will place a hat on each of your heads, and keep the remaining one to myself. Starting from the back, you must each guess the colour of your hat out loud and say nothing more. If you guess wrong, you shall become part of my breakfast. If you are right, I promise I won’t eat you until at least the following day. Now, off with you!”

With that, the dragon swept up the gnomes in one massive paw and tossed them into the cave, into which the other prisoners were already retreating.

“Sixteen colours!” groaned one of the gnomes. “How will we get out of this one? We barely managed with two!”

As they recovered from the shock of their capture, the ten gnomes began to take in their new surroundings. The light from the lava outside the cave mouth lit the interior with a dull red glow. The floor was littered with excrement and old bones, the remnants of animals occasionally given to the prisoners for food, or so the gnomes fervently hoped. At the back of the cave lay a large pile of rags, on which three of their fellow prisoners huddled together in misery.

Most remarkable of all, however, were the markings that covered the walls of the cave. Scratched in the rock were rows upon rows of diagrams, figures, mathematical formulae, etc, which the gnomes guessed to be attempted solutions to past problems set by the dragon. In one corner sat the remaining two prisoners, an old man and a young girl. The old man was hurriedly scratchimg into the wall a depiction of the problem the dragon had just posed, while the girl looked on, making the occasional quiet remark as the man wrote.

The most mathematically inclined gnome wandered over to this pair and joined in the discussion, while the remaining gnomes joined the other prisoners at the back of the cave. After some time contemplating their dismal fate, they heard a loud exclamation from the others, and looked to see the girl speaking excitedly and writing rapidly on the wall.

“It’s a simple parity argument,” the girl was saying. “First, we number the colours from 1 to 16. The configuration of hats can then be represented by a
permutation like so: the colour of the hat kept by the dragon Gaurinth gives the first number, the hat worn by the prisoner at the back gives the second, the hat worn by the next prisoner in line the third, and so on. For example, if Gaurinth keeps the hat with colour number 1, places hat with colour 2 on the prisoner at the back, the colour 3 hat on the next prisoner, and so on up to the colour 16 hat on the prisoner at the front, we can represent this by the permutation $\langle 1, 2, 3, \ldots , 16 \rangle$.

“Now, the prisoner at the back knows all the hat colours except for their own and Gaurinth’s—that is, all terms of the permutation except for the first two. So in our example, they know that the permutation is either $\langle 1, 2, 3, \ldots , 16 \rangle$ or $\langle 2, 1, 3, \ldots , 16 \rangle$, so their hat is coloured either 1 or 2. Somehow, we need their choice of 1 or 2 to give the rest of us enough information to figure out the colours of our hats.

“This is where parity comes in. Recall that a permutation has even parity if it can be obtained from $\langle 1, 2, 3, \ldots , 16 \rangle$ by swapping two entries an even number of times, and odd parity if not. So $\langle 1, 2, 3, \ldots , 16 \rangle$ has even parity (0 transpositions), $\langle 2, 1, 3, \ldots , 16 \rangle$ has odd parity (1 transposition—swap 1 and 2), $\langle 2, 3, 1, \ldots , 16 \rangle$ has even parity (2 transpositions—swap 1 and 2, then swap 1 and 3) and so on.

“The strategy we should use is as follows. Of the two choices of permutation the prisoner at the back has, one will be even and one will be odd. They should pick the even one, and guess their hat colour accordingly—hopefully they will be right. Now the next prisoner will hear this, and know all the terms of the even permutation except for the first and third, i.e., the colours of the hats in front of them, plus the one guessed behind them, but not their own or Gaurinth’s. This gives two choices, only one of which is even, so they can figure out the even permutation chosen by the prisoner behind exactly, and hence their own hat colour. Continuing like this, each of the remaining prisoners can work out the colour of their own hat!”

Once it was explained to them a few more times, the dragon’s prisoners all agreed this was a good solution, and the best they could hope for—in fact, better than most of them had hoped. They spent the rest of the night learning to calculate parities in their heads, to avoid any fatal mistakes the following day.

In the dark cavern in which they were trapped, the morning was heralded not by the coming of the sun, but by Gaurinth’s deep earth-shaking voice calling
“Come out of your hole, my morsels! I have your hats, and I grow hungry.”

The prisoners reluctantly moved out onto the ledge between the cave mouth and the lake of lava keeping them from escaping into the main body of Gaurinth’s lair. They saw that Gaurinth, who sat menacingly on his bed of gold, had laid out before him a row of sixteen different coloured hats. The prisoners quickly agreed on a numbering for the colours, before Gaurinth removed the hats, lined up his prisoners and began the trial.

The strategy was carried out effectively, luckily with no mistakes. Unfortunately, and to the great horror of the hindmost prisoner, the permutation Gaurinth had chosen was odd, not even, and the man was consumed in a snap of the dragon’s massive jaws.

Now Gaurinth was growing a little tired of this group of prisoners, especially the old man and young girl who had solved most of his problems since he had captured them while ransacking a university library in search of some decent books. He considered simply eating them without giving them a puzzle first, but this seemed unsporting—even the most wicked of dragons still have some twisted notion of fairness—as well as not being much fun. As he considered this, the dragon recalled a book he had been reading on set theory and was struck with an idea for a singularly diabolical puzzle.

“Very good!” boomed the dragon. “A reasonable display of ingenuity and mathematical knowledge. But enough of these child’s games; why don’t we have some real fun! I have another puzzle for you, but since this one is a little harder, I’ve decided that those of you who are successful may go free!”

At this the prisoners, some of whom had been without hope for months, began to listen with renewed interest.

“The problem is thus! I have in my lair 14 small caves. Into each of these, I will place an infinite number of boxes, labelled by the natural numbers 1, 2, 3, … and so on. In each box I will place a single real number, so that for every $n$, the number in box $n$ is the same in every room. You are allowed to confer together, before I place you one in each of the rooms, after which all communication will be impossible. In your separate rooms, you may each open as many boxes as you like—possibly infinitely many—as long as at least one box remains unopened. You must then choose an unopened box and guess the number within. If you are correct, you may go free! If not, then I shall eat
you.”

“Now, go back to your cave: you have until this evening to decide upon a strategy!”

As they staggered back into their cave, the prisoners stared at each other in horror. How could they possibly have any chance of solving this puzzle? How could the real number in an unopened box be determined from the opened ones? Surely Gaurinth was merely playing with them, and there was no solution to this puzzle at all!

One prisoner, however, was not so sure. The old man gazed thoughtfully into space before wandering over to a small patch of wall which still remained relatively clear and began to write. After a short while, he stopped and began to muse out loud.

“This reminds me of an old paradox in measure theory... But alas, it relied on the axiom of choice, which would be impossible to implement in practice!”

At this, the gnomes pricked up their ears. For gnomes are blessed with two magical abilities: the ability to stand very still for very long periods of time—an ability which has often caused them to be mistaken for garden ornaments—and the ability to make completely arbitrary decisions—even infinitely many of them!

After this was explained to him, the old man grinned. “Well, in that case, I think we might have a chance!”

To be continued...

— Dougal Davis

Paradox would like to thank Ruwan Devasurendra, Dougal Davis, Tien Hyung, Jason Tang and Sarah Boland for their contributions.