GAUSS'S EGREGIOUS THEOREM

MARTY ROSS

TODAY'S LECTURE
WILL BE IN MIME

THEOREMA EGREGIUM (1828)

GAUSS

"EGREGIOUS" - CONSPICUOUSLY BAD, OFFENSIVE

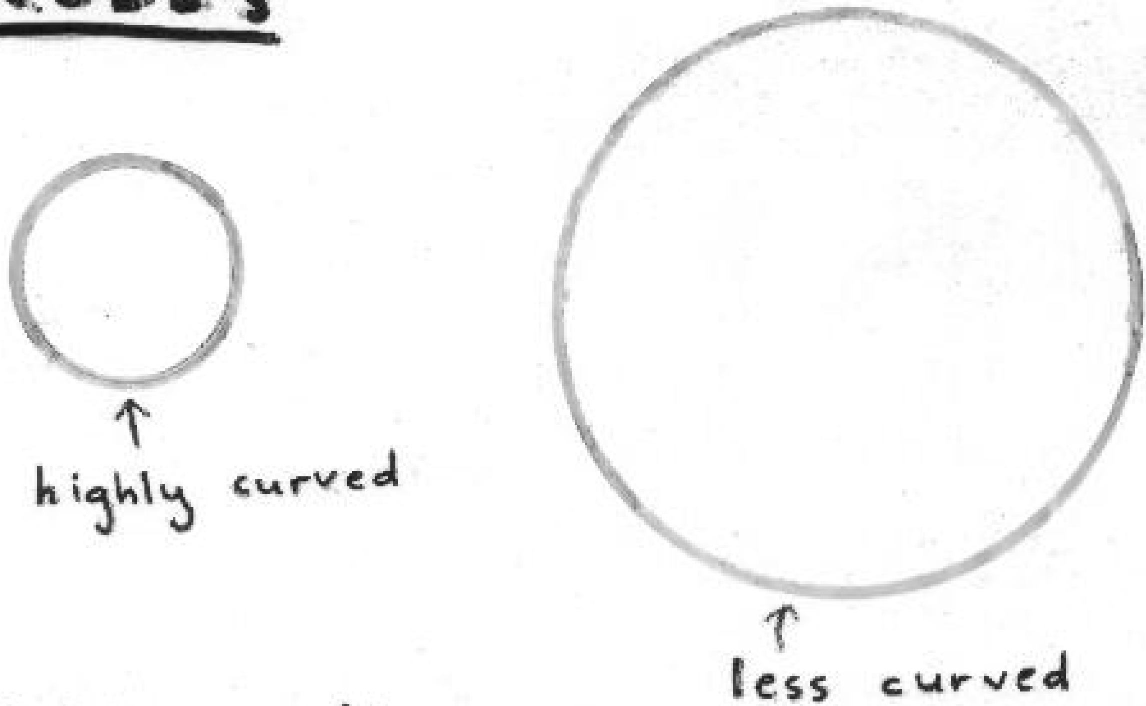
(LATIN - OUTSTANDING, PICK OF THE FLOCK)

REFERENCE:

"RIEMANNIAN GEOMETRY: A BEGINNER'S GUIDE"

FRANK MORGAN (233? OTHER?)

CIRCLES



Larger radius $\Rightarrow$ less curved

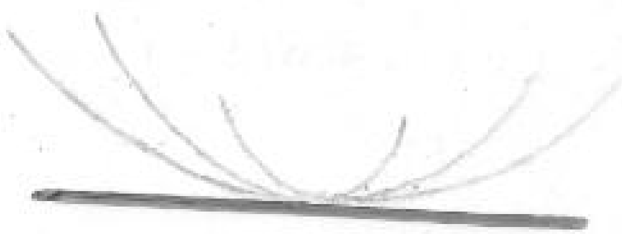
$$\Downarrow$$

Definition

The **curvature** of a circle of radius R is

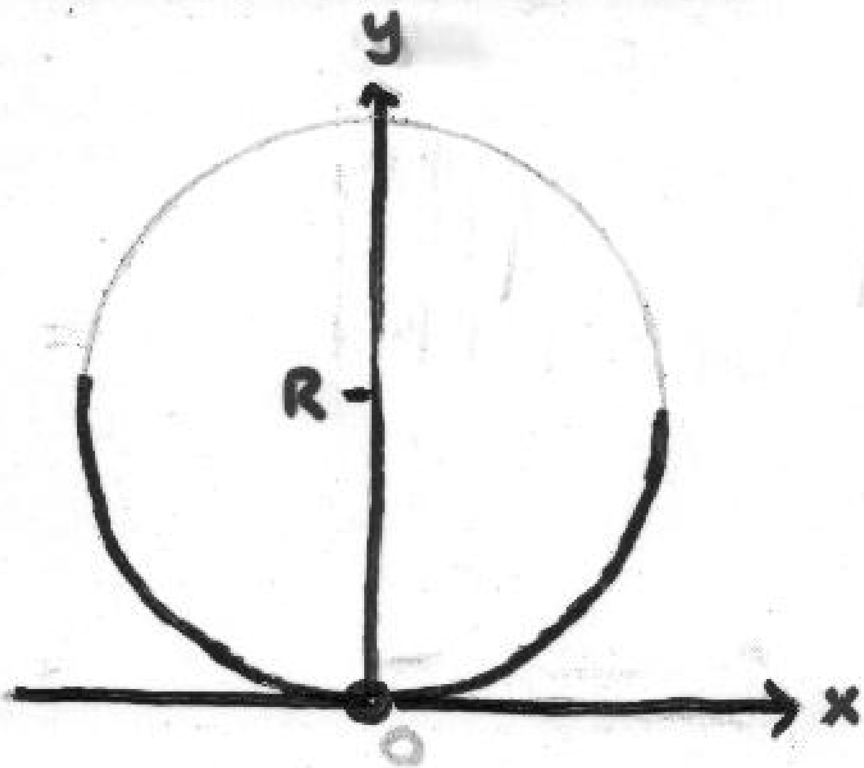
$$\boxed{k = 1/R}$$

Straight Lines



$R = \infty \Rightarrow k = 0.$

Circles as Graphs



$$(y-R)^2 + x^2 = R^2$$

$$y = R - \sqrt{R^2 - x^2}$$

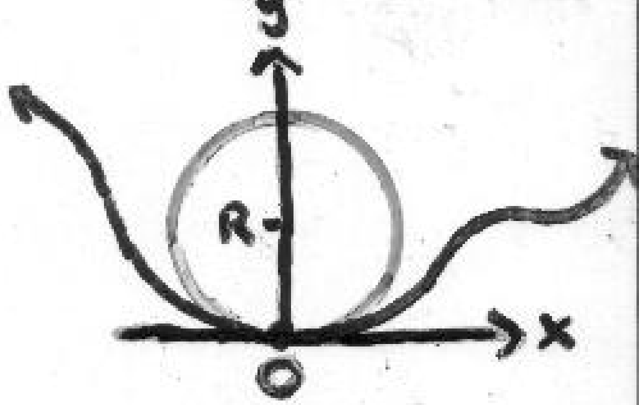
$$= R - R\sqrt{1 - x^2/R^2}$$

Taylor's Theorem

$$y \approx \frac{1}{2R}x^2 + \frac{1}{4R^3}x^4 + \cdots\cdots \qquad x \approx 0$$

$$\Rightarrow \boxed{y \approx \frac{k}{2}x^2 + \cdots\cdots}$$

General Graphs

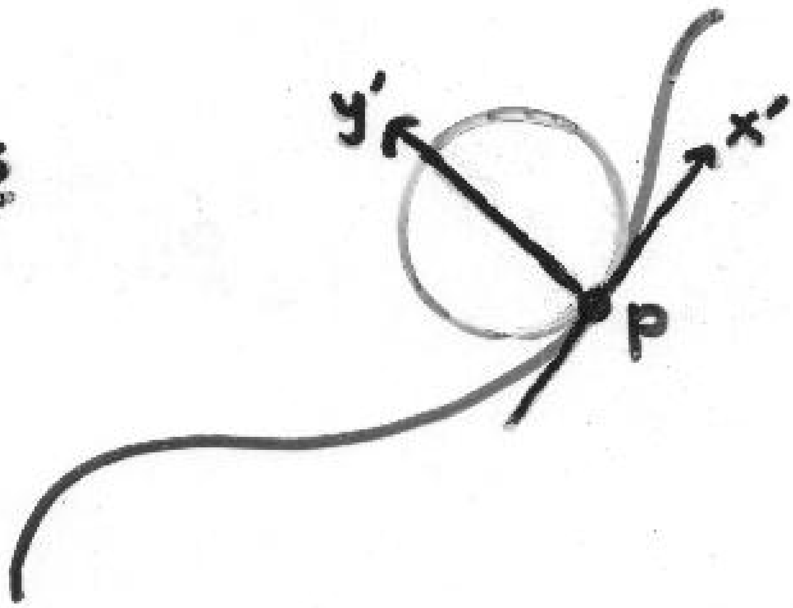


$y \approx A x^2 + \cdots\cdots$

$\Rightarrow$ $\boxed{k = 2A}$ at O (with $\frac{dy}{dx} = 0$)

"Circle of Best Fit" (Can be negative!)

At other Points



$y' \approx A(x')^2 + \cdots\cdots\cdots$ near p

$\Downarrow$

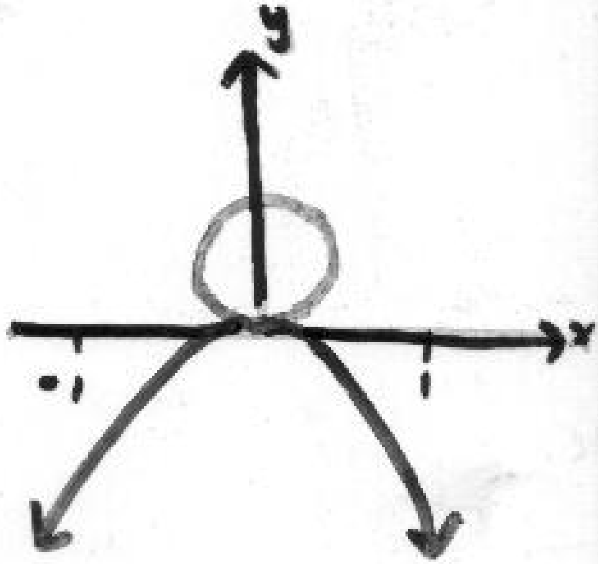
$\boxed{k(p) = 2A}$

Curvature is "Local"

Example

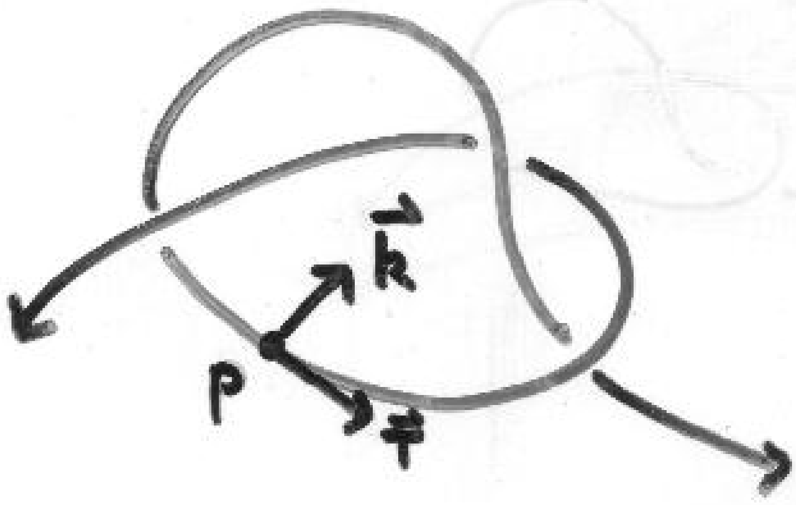
$$y = 1 \cdot x^2 - 10000\, x^4$$

$$\Rightarrow k(0) = 2$$



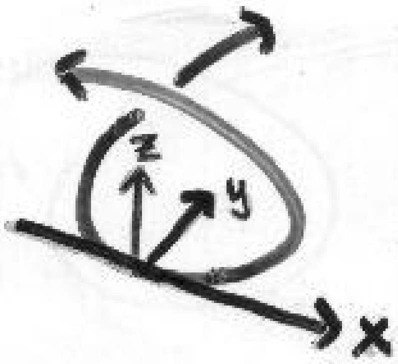
CURVATURE DEPENDS ONLY UPON 2nd DERIVATIVES

Space Curves



In suitable coordinates

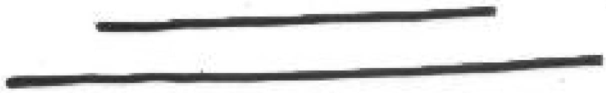
$$\begin{cases} y = Ax^2 + Bx^3 + \cdots \\ z = \qquad\quad Cx^3 + \cdots \end{cases}$$



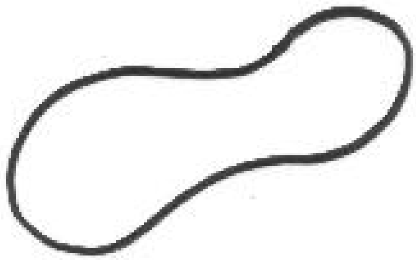
$$\boxed{\vec{k} = 2A\vec{j}}$$

WHAT CAN PATH-BUGS MEASURE?

LENGTH



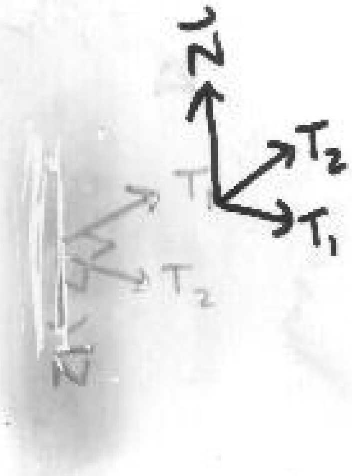
TOPOLOGY

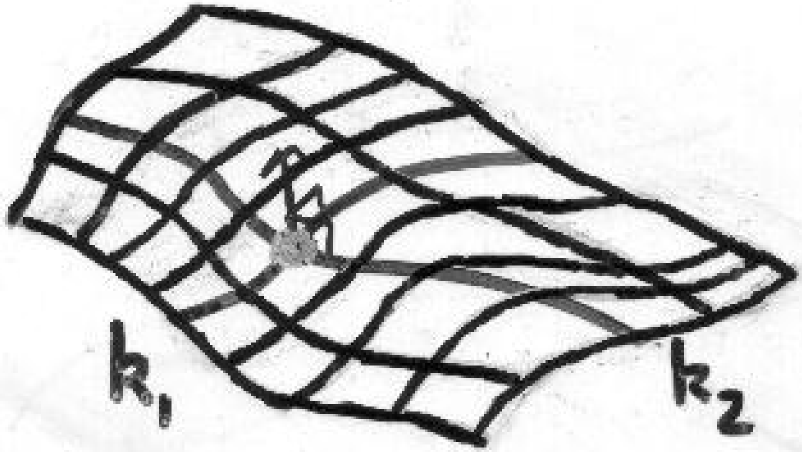


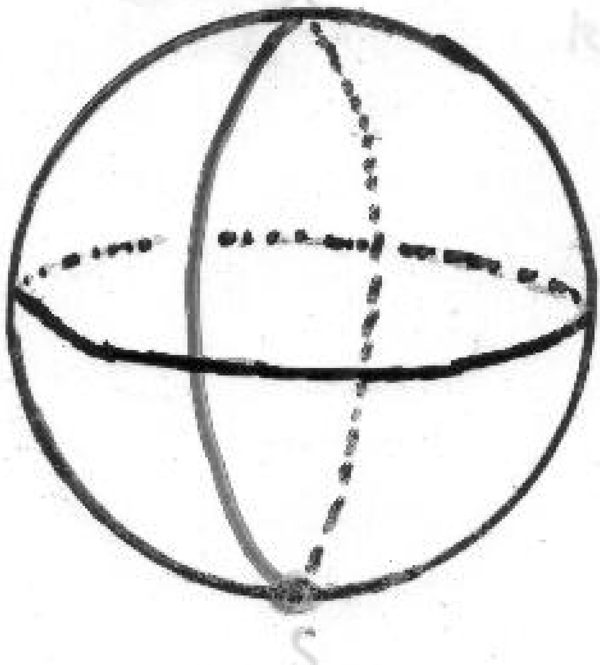
THAT'S IT!

K IS A GOD-QUANTITY

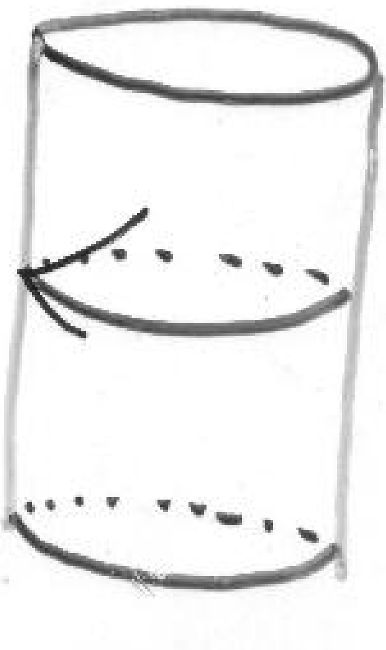
Surfaces







$k_1 = k_2 = 1/R$

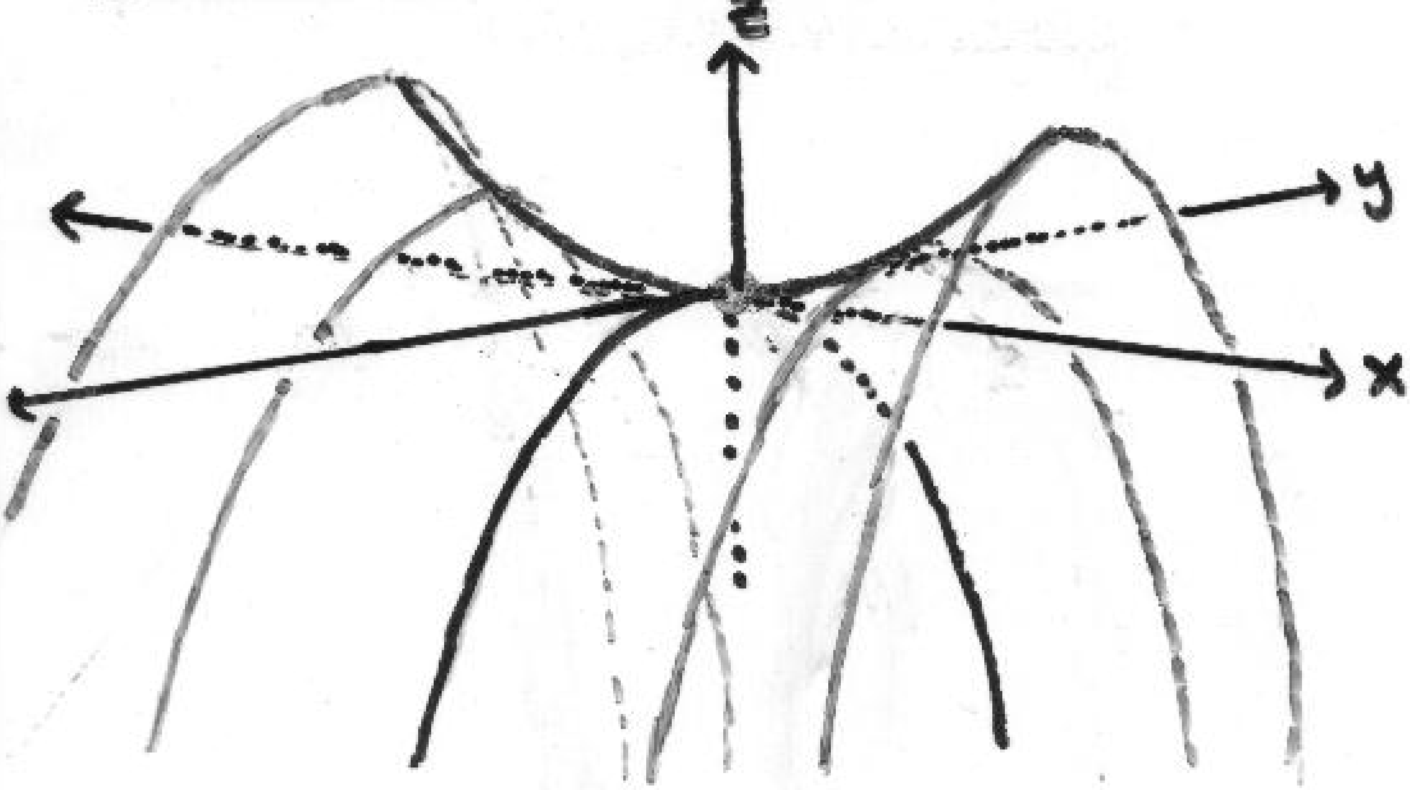


$k_1 = 1/R$

$k_2 = 0$

OTHER PARAMETERS?

Curvature of Surfaces



$$z = Ax^2 \qquad + Cy^2 + \cdots\cdots$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$\boxed{k_1 = 2A} \qquad \boxed{k_2 = 2C}$$

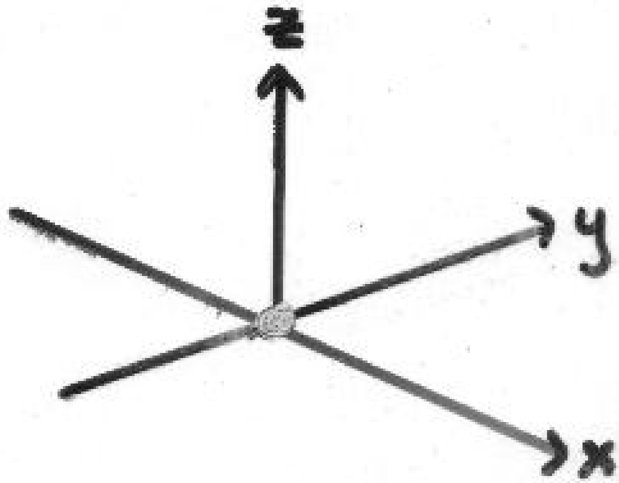
(CAN BE NEGATIVE)

$$\boxed{H = k_1 + k_2}$$ Mean Curvature

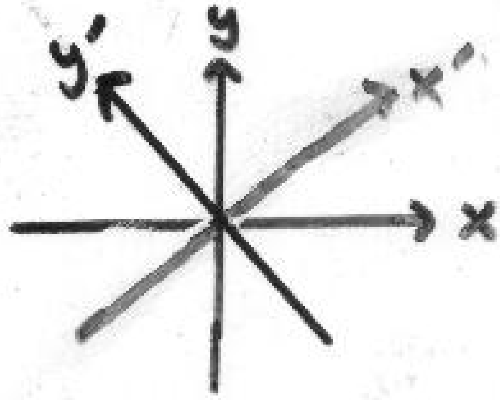
$$\boxed{K = k_1 \cdot k_2}$$ Gauss Curvature

Example

$$z = xy$$



$$\begin{cases} y = 0 \Rightarrow z = 0 \\ x = 0 \Rightarrow z = 0 \end{cases}$$



Rotate by $\pi/4$ around z

$$\begin{cases} x = \frac{1}{\sqrt{2}} x' - \frac{1}{\sqrt{2}} y' \\ y = \frac{1}{\sqrt{2}} x' + \frac{1}{\sqrt{2}} y' \end{cases}$$

$$\Downarrow$$

$$z = \tfrac{1}{2}(x')^2 - \tfrac{1}{2}(y')^2$$

$$\Downarrow$$

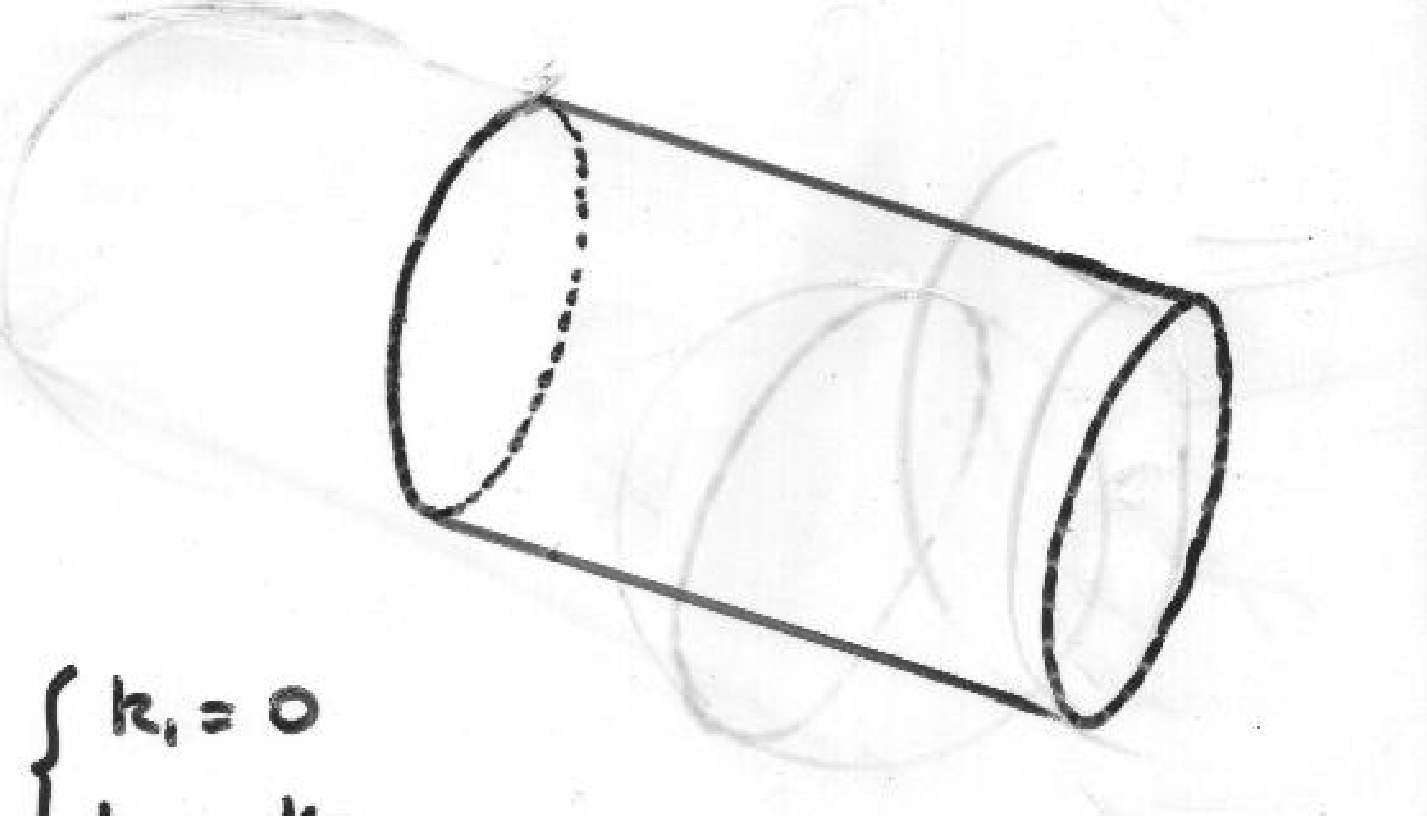
$$\boxed{\begin{array}{c} H = 0 \\ K = -1 \end{array}}$$

$$z = ax^2 + bxy + cy^2 + \cdots$$

$$\Downarrow \text{ rotate}$$

$$z = A(x')^2 \qquad + C(y')^2 + \cdots$$

Cylinders



$$\begin{cases} k_1 = 0 \\ k_2 = 1/R \end{cases}$$

$$\Downarrow$$

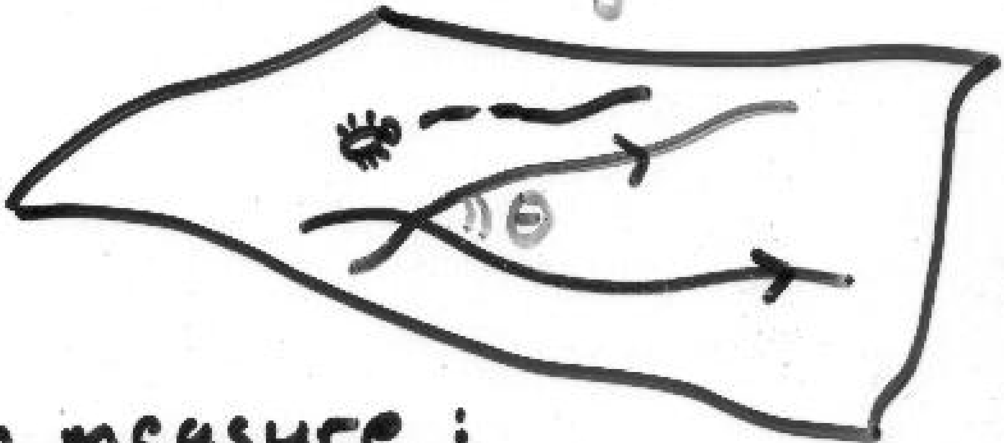
$$\boxed{\begin{aligned} H &= 1/R \\ K &= 0 \end{aligned}}$$

Spheres

$$\boxed{\begin{aligned} H &= 2/R \\ K &= 1/R^2 \end{aligned}}$$

BUGMATHS

Intrinsic Geometry



Bugs can measure:

(*) LENGTHS of paths

(*) ANGLES between paths

(*) WHAT ELSE?

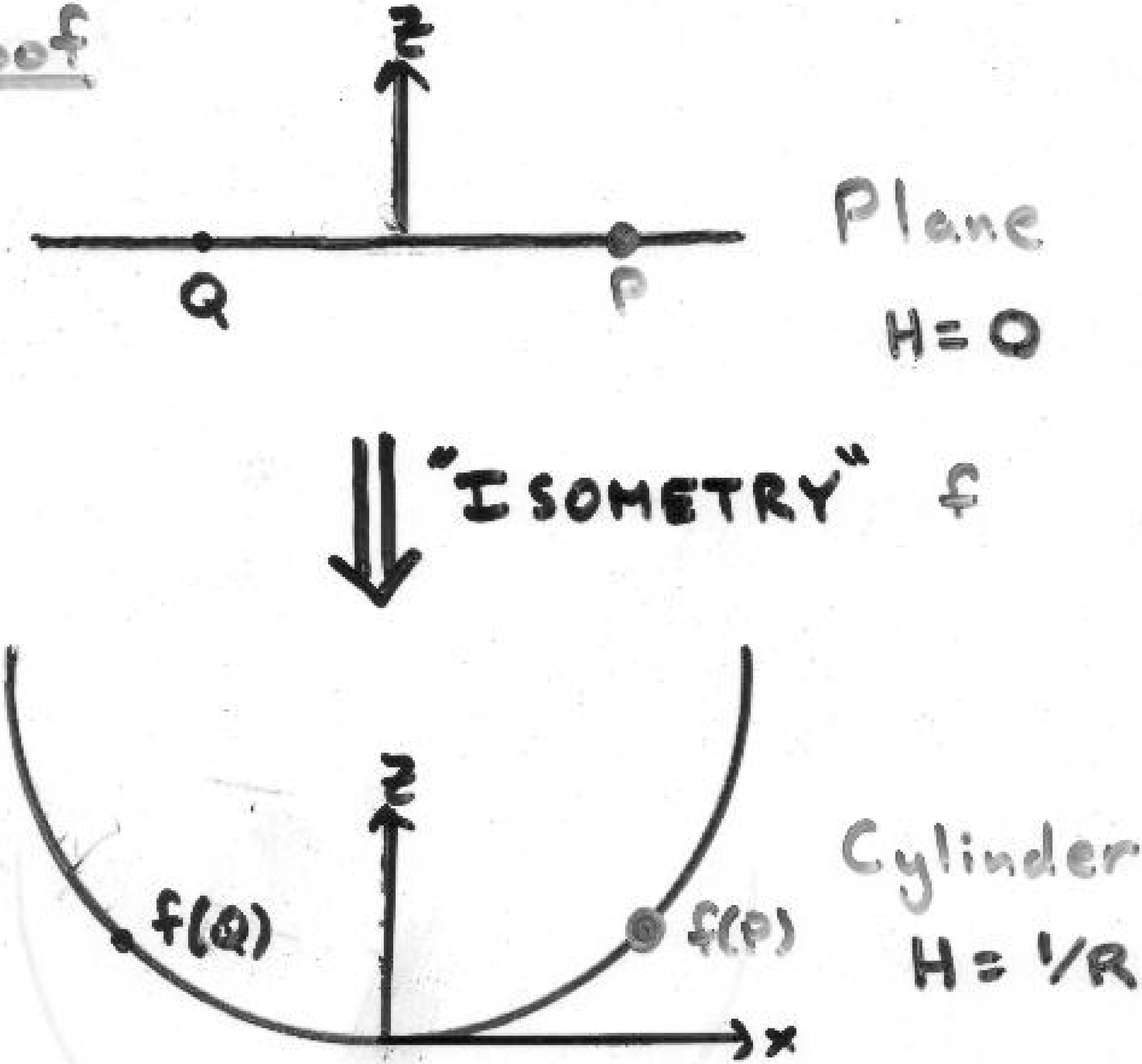
AREAS of regions

CURVATURE?

UnRemarkable Theorem

Bugs CANNOT measure mean curvature. H.

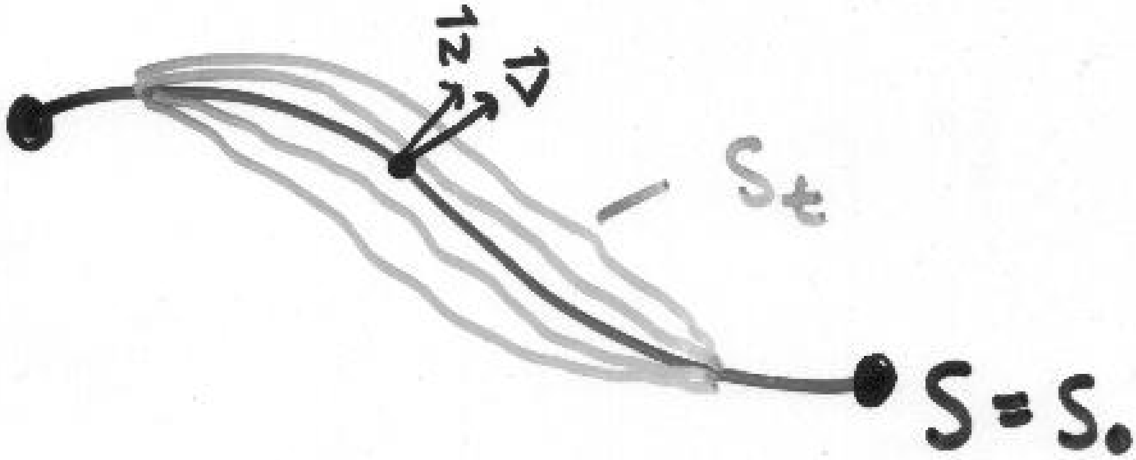
Proof



$$f(x,y) = (R\sin x/R,\ y,\ R(1-\cos x/R))$$

(dist(f(P), f(Q)) = dist(P,Q))

INTERLUDE: THE MEANING OF H=0.



PERTURB A SURFACE S

$A(t) = \text{Area}(S_t)$

THEN

$$\boxed{A'(0) = \int_S H\, \vec{N}\cdot\vec{V}\, dA}$$

So $H=0 \iff$ S satisfies a first derivative test to have least area.

"Minimal Surfaces"
Soap Films

Gauss's Theorema Egregium

Bugs CAN measure Gauss curvature!

Corollary

If $K \neq 0$ for a surface S then any map of S is distorted.

Gauss's Proof of T.E.

In suitable bug-coordinates p, q

$$K = \frac{\partial^2 F}{\partial p \partial q} - \frac{1}{2}\frac{\partial^2 G}{\partial p^2} - \frac{1}{2}\frac{\partial^2 E}{\partial q^2}$$

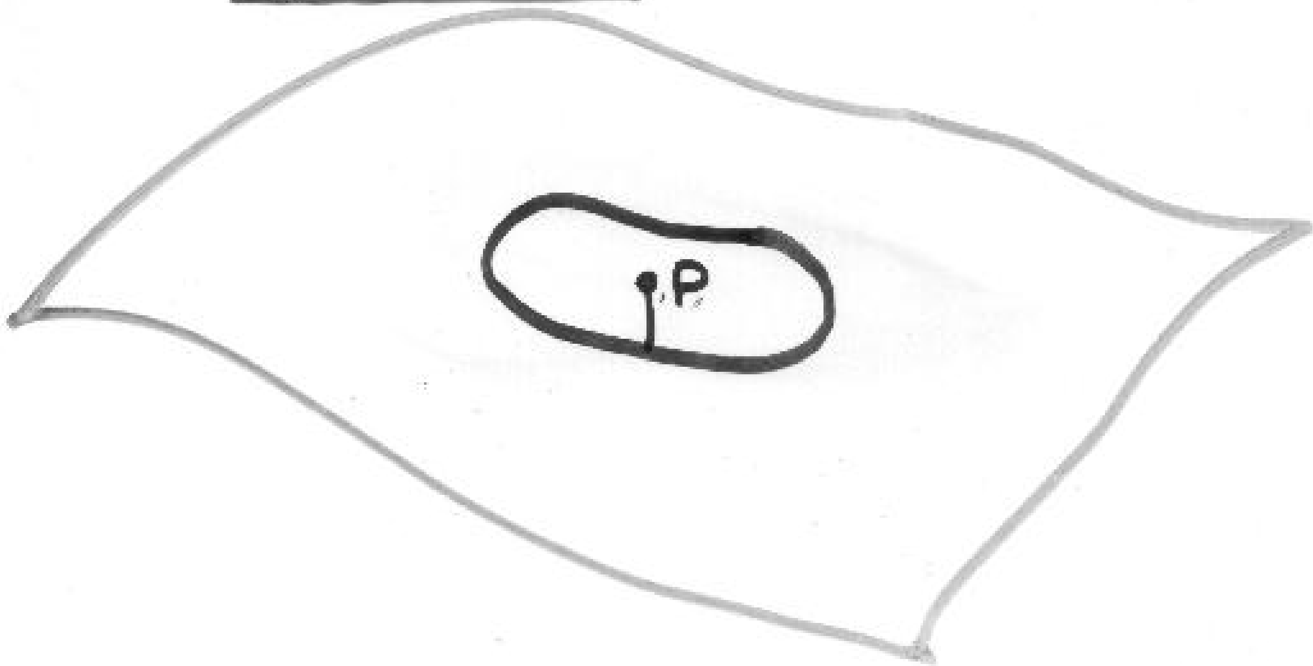
= explicit expression in terms of path quantities.

Geometric Interpretation Proof ...

Capsule History

NEWTON	1665	Curvature of paths
Euler	1760	k_1 and k_2
	1772	"Developable surfaces"
	1775	Maps of spheres are distorted
Meusnier	1776	Mean curvature H
Gauss	1818	Survey of Hannover
	1828	K and $\boxed{\text{Theorema Egregium}}$
Minding ?	1839 ?	$K_1 = K_2 \Rightarrow$ undistorted map.

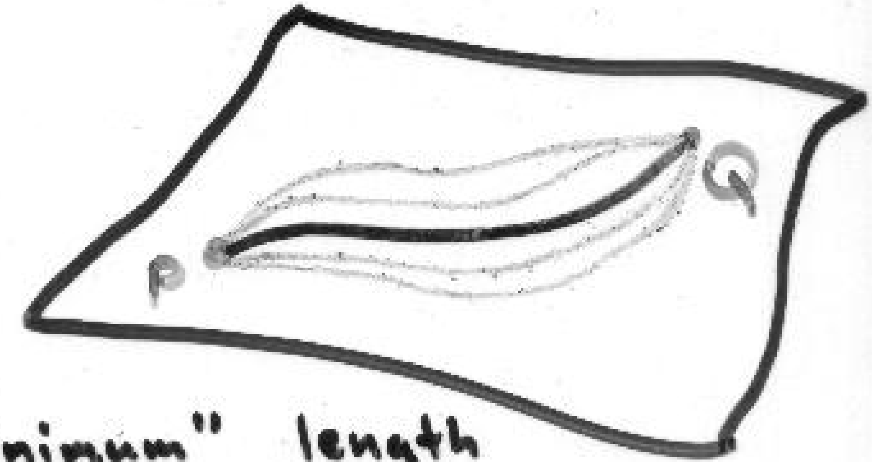
QUESTION



WHAT'S THE AREA OF A DISK $D_r(P)$ OF RADIUS R?

what's a "DISK"?

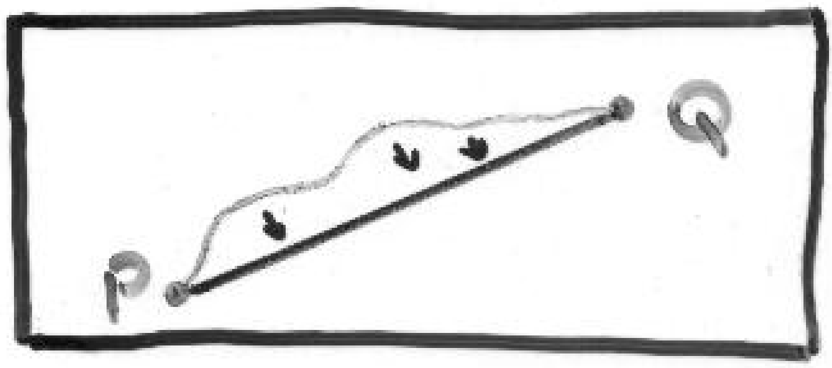
BUG-DISTANCE



$b(P,Q)$ = the "minimum" length of surface-paths from P to Q.

Geodesic = a path of (local) minimum length

Plane

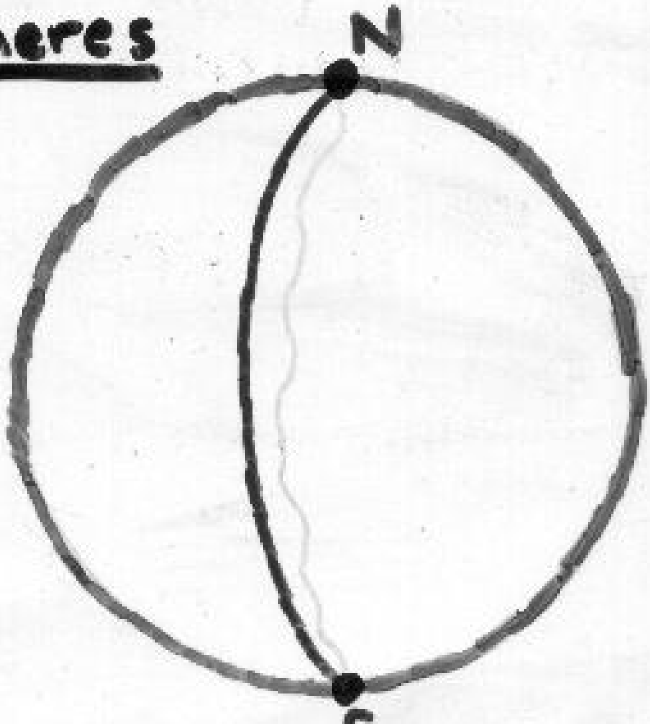


$b(P,Q) = d(P,Q) =$ Euclidean distance

In general $\boxed{b(P,Q) \geq d(P,Q)}$

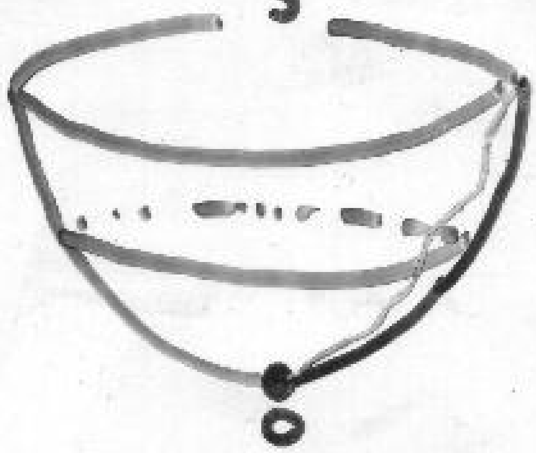
Geodesics on Spheres

"Great Circles"

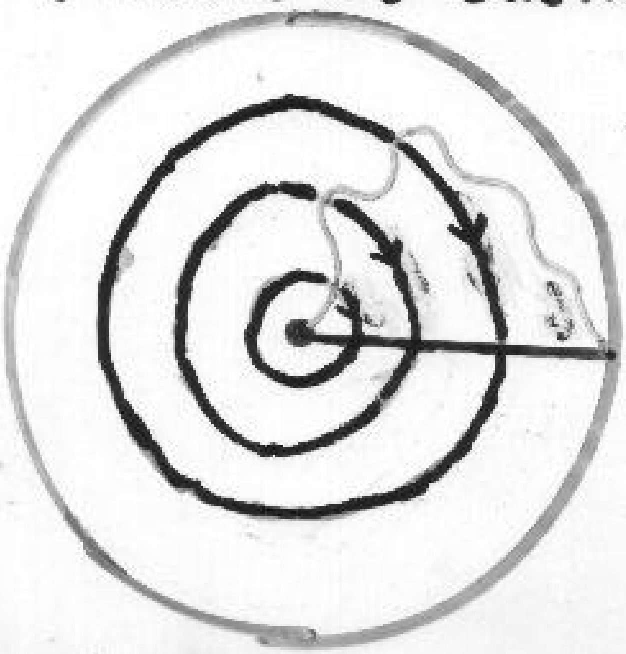


Theorem

If S is a surface rotationally symmetric about O then the geodesics leaving O are the radial lines.

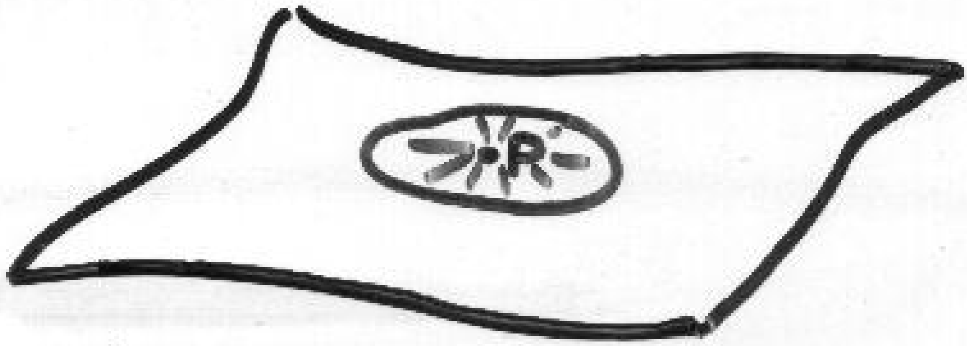


Proof (Same as Euclidean)



"projection" is length-decreasing.

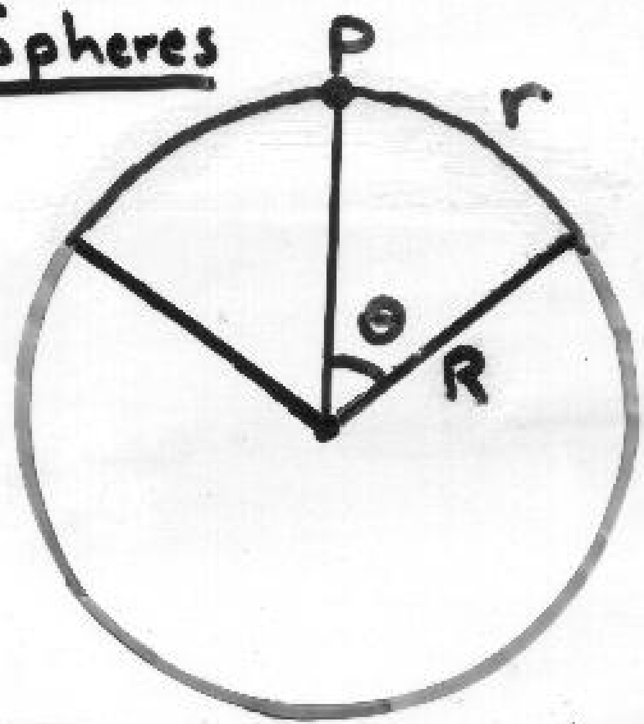
Geodesic Disks



$D_r(P) =$ the points within distance r of P.

Euclidean Disks $\quad \mathrm{Area}(D_r(P)) = \pi r^2$

Disks on Spheres



$\theta = r/R$

$$\begin{aligned} \mathrm{Area} &= 2\pi R^2 (1 - \cos(r/R)) \\ &= \pi r^2 \left(1 - \frac{1}{12 R^2} r^2 + \dots\right) \\ &= \boxed{\pi r^2 \left(1 - \frac{K}{12} r^2 + \dots\right)} \end{aligned}$$

$\Rightarrow$ Maps of spheres are distorted.

Disk Theorem (Gauss?) (Minding?)

On any surface

$$\text{area}(D_r(P)) = \pi r^2\left(1 - \frac{K}{12} r^2 + \cdots\right)$$

$K(P)$

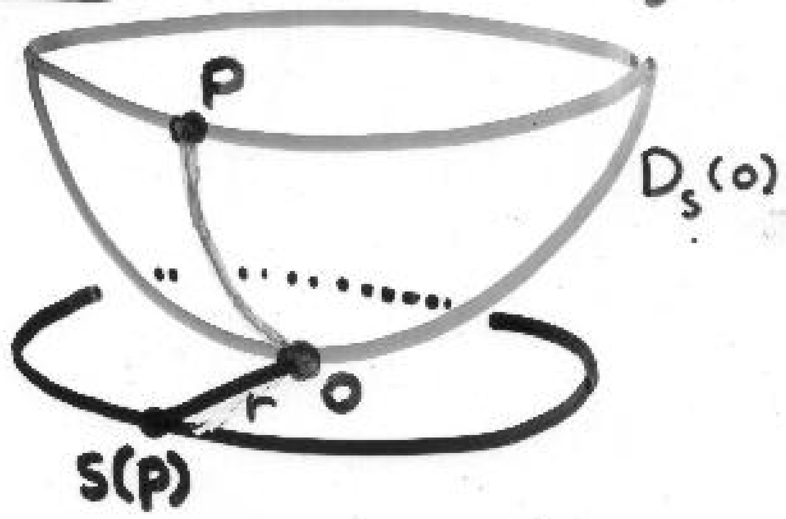
r small

$$\Downarrow$$

Theorema Egregium is immediate.

Shadows

$$z = Ax^2 + Cy^2 + \cdots\cdots$$



$\ell(P)$ = length of "radial path" from O to P.

$$= r + \frac{2r^3}{3}\left(A\cos^2\theta + C\sin^2\theta\right)^2 + \cdots\cdots$$

(Polar Coordinates)

By definition,

$$\ell(P) \geq b(o,P) = s$$

Key Fact Radial Paths are almost geodesics:

$$\ell(P) \leq b(o,P) + Mr^4$$

constant depending upon A, C.

(this takes thought!)

$$s = b(0, P) \approx l(P) \cong r + \frac{2r^3}{3}\left(A\cos^2\theta + C\sin^2\theta\right)^2$$

Consequently

The shadow of $D_s(0)$ is approximately the region defined in polar coordinates by the equation

$$\boxed{r = s - \frac{2s^3}{3}\left(A\cos^2\theta + C\sin^2\theta\right)^2}$$

Consequently

We can calculate the approximate area of $D_s(0)$

$$\Downarrow$$

$$A = \pi s^2\left(1 - \frac{AC}{3}s^2 + \cdots\right)$$
$$= \pi s^2\left(1 - K/_{12}\, s^2 + \cdots\right)$$