Keller's model of long distance running

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How should a runner vary his speed $v(t)$ during a race of distance D in order to minimise his time T? Then

$$D = \int_0^T v(t)dt. \tag{1}$$

Since $F = ma$

$$\frac{dv}{dt} + h(v) = \frac{F}{m} = f(t), \tag{2}$$

where $f(t) = F/m$, the propulsive force per unit mass. $h(v)$ is a resistive force. First air resistance, experimentally determined as $f = kv^2$, with $k = 0.00035$ kg/kg. At $10m/s$ this is 35gm/kgm, so fairly negligible. More significantly, there is a force proportional to v due to frictional forces, muscle viscosity, stretching of elastic structures (remember Hooke $\ddot{x} = -cx$.)

So (2) becomes

$$\frac{dv}{dt} + \frac{v}{\tau} = f(t) \tag{3}$$

where τ is to be determined from data. Initially,

$$v(0) = 0 \tag{4}$$

and $f(t)$ is controlled by the runner, but

$$0 \leq f(t) \leq F. \tag{5}$$

Now, force f is controlled by amount of fuel, in this case oxygen. Let $E(t)$ be available oxygen/unit mass in muscles. Assume this is consumed in energy releasing reactions. Now, rate of doing work (fv)/unit mass = rate at which body supplies energy. Since $E = mv^2/2$, $\frac{\mathrm{d}E}{\mathrm{d}t} = mv\frac{\mathrm{d}v}{\mathrm{d}t} = mva$.

When one runs, the cardiovascular system supplies oxygen. Breathing harder, and higher pulse produces more oxygen than at rest. Let σ be the excess rate of oxygen production. Then we have an oxygen balance equation, where we measure oxygen in units of the amount of energy it can yield in reaction.

$$\frac{dE}{dt} = \sigma - fv. \tag{6}$$

Note that

$$E(0) = E_0 \geq 0 \tag{7}$$

and

$$E(t) \geq 0 \tag{8}$$

Problem: find $v(t)$, $f(t)$ and $E(t)$ satisfying eqns. (2)-(8) so that T in (1) is minimised. Assume the 4 physiological constants τ, F, σ and E_0 are given, as is T.

This is a calculus of variations problem with two d.e.s and 3 unknown functions. Hence an infinity of solutions. We want the optimal solution. This is a problem in optimal control theory. The force $f(t)$ is the control variable. We first reformulate by expressing both $f(t)$ and $E(t)$ in terms of $v(t)$. Eqn. (3) + (5) gives

$$\frac{dv}{dt} + \frac{v}{\tau} \leq F. \tag{9}$$

Integrating eqn. (6) from 0 to T gives

$$E(T) - E_0 = \sigma T - \frac{1}{2}v^2(T) - \frac{1}{\tau}\int_0^T v^2(t)dt \tag{10}$$

Hence

$$E_0 + \sigma T - \frac{v^2(T)}{2} - \frac{1}{\tau}\int_0^T v^2(t)dt \geq 0. \tag{11}$$

Now eqns. (4), (9) and (11) are our key equations. We will maximise D with T fixed. Initially we can make f as large as we like ($\leq F$ of course.) Assume we do this for an initial burst, so $f(t) = F$ for $0 \leq t \leq t_1$. (If $t_1 \leq 0$, then this will be nonsense). So

$$\frac{dv}{dt} + \frac{v}{\tau} = F \qquad 0 \leq t \leq t_1 \tag{12}$$

with solution (simple, separable o.d.e.).

$$v(t) = F\tau\left(1 - e^{\frac{-t}{\tau}}\right) \qquad 0 \leq t \leq t_1 \tag{13}$$

Substitute into eqn. (11), gives

$$E_0 + \sigma t - F^2 t^2 \left(\frac{t}{\tau} + e^{\frac{-t}{\tau}} - 1 \right) \geq 0 \qquad 0 \leq t \leq t_1 \tag{14}$$

If $\sigma \geq F^2\tau$, (14) is satisfied for all $t \geq 0$. Clearly unrealistic—it means sprint flat out. So consider the case $\sigma < F^2\tau$. Then (14) holds for $0 \leq t \leq T_c$, where T_c is the unique positive root of (14) with equality holding. The inequality holds for $t < T_c$, and in this case (13) yields the optimal sprint velocity, with

$$D = F\tau^2 \left(\frac{T}{\tau} + e^{\frac{-T}{\tau}} - 1 \right) \qquad 0 \leq T \leq T_c \tag{15}$$

the length of the longest possible sprint D_c given by (15) with $T = T_c$. If $T > T_c$, then $t_1 < T$ and we must find t_1 and the function $v(t)$ for $t_1 < t < T$.

At the end of the race we require $E(T) = 0$—all oxygen used up. Let's go further, and assume $E(t) = 0$ throughout the interval $t_2 \leq t \leq T$. That is run the last little bit without oxygen, just let kinetic energy carry you. Thus we assume that for some $t_2 \geq t_1$ we have

$$E(t) = 0 \qquad t_2 \leq t \leq T. \tag{16}$$

Then (10) becomes

$$0 = E_0 + \sigma t - \frac{1}{2} v^2(t) - \frac{1}{\tau} \int_0^t v^2(s) ds \qquad t_2 \leq t \leq T \tag{17}$$

Then differentiating this w.r.t. t gives

$$\sigma - \frac{1}{2} \left(\frac{dv^2(t)}{dt} \right) - \frac{v^2(t)}{\tau} = 0 \qquad t_2 \leq t \leq T. \tag{18}$$

Solving this gives

$$v^2(t) = \sigma\tau + \left(v^2(t_2) - \sigma\tau \right) e^{\frac{2(t_2 - t)}{\tau}} \qquad t_2 \leq t \leq T. \tag{19}$$

Note that neither has $v(t_2)$ been determined, nor (16) satisfied. Instead, we set $dE/dt = 0$, so (16) will hold if we set $E(t_2) = 0$. We need to determine $v(t)$ for $t_1 \leq t \leq t_2$ and $v(t_2)$. We have found $v(t)$ to be given by (13) for the first few hundred metres $(0 \leq t \leq t_1)$, and by (19) in the last few metres $(t_2 \leq t \leq T.)$ We need $v(t)$ in the interval $t_1 \leq t \leq t_2$. Re-write (1)

$$D = \int_0^{t_1} F\tau \left(1 - e^{\frac{-t}{\tau}}\right) dt + \int_{t_1}^{t_2} v(t)dt + \int_{t_2}^{T} \sqrt{\sigma t + \left(v^2(t_2) - \sigma\tau\right) e^{\frac{2(t_2 - t)}{\tau}}}\, dt \quad (20)$$

$D = D(v(t), t_2)$ is called a *functional*. We must choose $v(t)$ and t_2 to maximise D subject to

$$E(t_2) = 0 \quad (21)$$

which we do by the calculus of variations.

We wish to extremise $f(x, y)$, subject to constraint $g(x, y)$, yielding x_0, y_0. First set

$$\phi = f(x, y) + \lambda g(x, y),$$

then $\frac{\partial \phi}{\partial x} = 0$ determines x_0, y_0 in terms of λ, and λ is found from the constraint. From (10) we can get

$$E(t_2) = E_0 + \sigma t_2 - \frac{1}{2} v^2(t_2) - \frac{1}{\tau} \int_0^{t_2} v^2(t) dt$$

and we want to maximise the functional D subject to $E(t_2) = 0$.

Let

$$\phi(v(t), v(t_2)) = D + \lambda E,$$

then

$$\frac{\partial \phi}{\partial v(t)} = 0 \qquad \frac{\partial \phi}{\partial v(t_2)} = 0. \tag{22}$$

The above equation can be written

$$0 = E(t_2) = E_0 + \sigma t_2 - \frac{1}{2}v^2(t_2) - \frac{1}{\tau}\int_0^{t_2} v^2(t)dt$$
$$= E_0 + \sigma t_2 - \frac{1}{2}v^2(t_2) - \frac{1}{\tau}\int_0^{t_1} F^2\tau^2\left(1 - e^{\frac{-t}{\tau}}\right)^2 dt - \frac{1}{\tau}\int_{t_1}^{t_2} v^2(t)dt \tag{23}$$

and from (22), with $\phi = D + \lambda E$

$$\frac{\partial \phi}{\partial v(t)} = \int_{t_1}^{t_2} dt + \lambda\left(-\frac{2}{\tau}\int_{t_1}^{t_2} v(t)dt\right) = \int_{t_1}^{t_2}\left(1 - \frac{2\lambda v}{\tau}\right)dt = 0 \tag{24}$$

which can only hold if the integrand vanishes, so

$$v = \frac{\tau}{2\lambda} \qquad t_1 \leq t \leq t_2. \tag{25}$$

The second equation of (22) gives

$$\frac{\partial \phi}{\partial v(t_2)} = v(t_2) \int_{t_2}^{T} \frac{e^{\frac{2(t_2-t)}{\tau}}}{\sqrt{\sigma\tau + \left(v^2(t_2) - \sigma\tau\right) e^{\frac{2(t_2-t)}{\tau}}}} dt - \lambda v(t_2) = 0. \tag{26}$$

Hence

$$\lambda = \int_{t_2}^{T} \frac{e^{\frac{2(t_2-t)}{\tau}}}{\sqrt{\sigma\tau + \left(v^2(t_2) - \sigma\tau\right) e^{\frac{2(t_2-t)}{\tau}}}} dt \qquad v(t_2) \neq 0 \tag{27}$$

and sustituting $v = \frac{\tau}{2\lambda}$

$$\lambda = \int_{t_2}^{T} \frac{e^{\frac{2(t_2-t)}{\tau}}}{\sqrt{\sigma\tau + \left(\frac{\tau^2}{4\lambda^2} - \sigma\tau\right) e^{\frac{2(t_2-t)}{\tau}}}} dt \tag{28}$$

which can be integrated to give

$$\lambda = \frac{\sqrt{\sigma\tau + \left(\frac{\tau^2}{4\lambda^2} - \sigma\tau\right) e^{\frac{2(t_2 - T)}{\tau}}} - \frac{\tau}{2\lambda}}{\sigma - \frac{\tau}{4\lambda^2}}. \tag{29}$$

From (22) and the condition $E(t_2) = 0$

$$\begin{aligned} 0 &= E_0 + \sigma t_2 - \frac{1}{2}\left(\frac{\tau^2}{4\lambda^2}\right) - \frac{1}{\tau}\int_0^{t_1} F^2\tau^2\left(1 - e^{\frac{-t}{\tau}}\right)^2 dt - \frac{1}{\tau}\int_{t_1}^{t_2}\left(\frac{\tau^2}{4\lambda^2}\right) dt \\ &= E_0 + \sigma t_2 - \frac{\tau^2}{8\lambda^2} - \frac{\tau}{4\lambda^2}(t_2 - t_1) - F^2\tau\left(t_1 + 2\tau e^{\frac{-t_1}{\tau}} - \frac{\tau}{2} e^{\frac{-2t_1}{\tau}} - \frac{3\tau}{2}\right). \end{aligned} \tag{30}$$

Hence

$$\lambda^2 = \frac{1}{8}\left(\frac{\tau^2 + 2\tau(t_2 - t_1)}{E_0 + \sigma t_2 - F^2\tau(t_1 + 2\tau e^{\frac{-t_1}{\tau}} - \frac{\tau}{2} e^{\frac{-2t_1}{\tau}} - \frac{3\tau}{2})}\right) \tag{31}$$

Now (13) is $v(t) = F\tau(1 - e^{\frac{-t}{\tau}})$ for $0 \leq t \leq t_1$, so

$$v(t_1) = F\tau\left(1 - e^{\frac{-t_1}{\tau}}\right) = \frac{\tau}{2\lambda} \tag{32}$$

and from (25)

$$v(t_1) = \frac{\tau}{2\lambda} = v(t_2). \tag{33}$$

So these few equations above determine t_1, t_2 and λ provided that $t_1 \leq t_2 \leq T$. Carefully examing the structure of the equation for λ shows there are four possible solutions,

$$2\lambda = \pm\sqrt{\frac{\tau}{\sigma}} \tag{34}$$

and

$$2\lambda = \pm\sqrt{\frac{\tau}{\sigma}} \cdot \sqrt{1 - 4e^{\frac{2(t_2 - T)}{\tau}}}. \tag{35}$$

The negative roots lead to unphysical solutions, while $2\lambda = \sqrt{\frac{\tau}{\sigma}}$ violates (28). Thus

$$2\lambda = \sqrt{\frac{\tau}{\sigma}} \cdot \sqrt{1 - 4e^{\frac{2(t_2 - T)}{\tau}}}. \tag{36}$$

Substitute into (32) and set $t_1 = t_2 = T_c$. That gives

$$\frac{\sqrt{\frac{\sigma}{\tau}}}{\sqrt{1 - 4e^{\frac{-2(T - T_c)}{\tau}}}} = F\left(1 - e^{\frac{-T_c}{\tau}}\right) \tag{37}$$

or

$$T_* = T_c + \tau\left(\log 2 - \frac{1}{2}\log\left(1 - \frac{\sigma}{F^2\tau\left(1 - e^{\frac{-T_c}{\tau}}\right)^2}\right)\right). \tag{38}$$

For longer distances, $T \geq T_*$ eqns. (29), (30) and (31) yield t_1, t_2 and T_c. For $T_c \leq T \leq T_*$ we have instead $t_1 = t_2 = T_c$ and (31) yields λ. Having found t_1, t_2 and λ, $v(t)$ is given by substitution into earlier equations. The distance travelled is

$$
\begin{aligned}
D = F\tau^2 \left(\frac{t_1}{\tau} + e^{\frac{-t_1}{\tau}} - 1 \right) + \frac{\tau(t_2 - t_1)}{2\lambda} + \\
\tau\sqrt{\sigma\tau} \left(-\frac{1}{2\lambda}\sqrt{\frac{\tau}{\sigma}} - \tanh^{-1}\left(\frac{1}{2\lambda}\sqrt{\frac{\tau}{\sigma}} \right) \right) \\
- \tau\sqrt{\sigma\tau} \left(\sqrt{1 + \left(\frac{\tau}{4\sigma\lambda^2} - 1 \right) e^{\frac{-2(T-t_2)}{\tau}}} \right) \\
+ \tau\sqrt{\sigma\tau} \left(\tanh^{-1} \sqrt{1 + \left(\frac{\tau}{4\sigma\lambda^2} - 1 \right) e^{\frac{-2(T-t_2)}{\tau}}} \right) \\
T \geq T_c. \quad (39)
\end{aligned}
$$

For $T \leq T_c$, $D = F\tau^2(T/\tau + e^{-T/\tau} - 1)$.

Remember, we assumed $f(t) = F$ in the initial interval $0 \le t \le t_1$ and $E(t) = 0$ in the interval $t_2 \le t \le T$.

The constants are obtained from published world records by

$$\text{Minimise} \sum \frac{(T_{\text{world record}} - T_{\text{calc.}})^2}{T^2_{\text{world record}}}$$

$$\tau = 0.892\,\text{sec},\ F = 12.2\,\text{m/sec}^2,\ \sigma = 9.93\,\text{cal/kg}\cdot\text{sec},\ E_0 = 575\,\text{cal/kg}. \quad (40)$$

It follows that $D_c = 291$m, consistent with the fact that 200m is a sprint and 400m isn't. Strategy for $D > D_c$ is to go flat-out to cruising speed, and hold that till the last 1 sec (400m) or 2 sec (10 km), then "fall" across the line.