620-362 Applied Operations Research

SAMPLE MID-SEMESTER TEST

Duration: 90 minutes (1.5 hours)

Total Marks: 100

This test comprises 7 questions and 10 pages.

This test must be your own work; no discussion or collaboration is permitted. Calculators, notes and books are *not* allowed to be used.

NAME:

Question	Marks Available	Marks Awarded
1	7	
2	17	
3	27	
4	15	
5	12	
6	7	
7	15	
Total	100	

Question 1.
[7 marks]

Consider the algorithm below for finding the factorial of a positive number `n`.

Line Number Line of Algorithm

```
1    fact := n;
2    for k = 1,...,n-1 do
3        fact := fact * (n - k);
4    endfor
```

(a) How many basic operations are required to execute this algorithm, as a function of `n`? Justify your answer briefly.

Solution: Line 1 requires 1 operation. To initialize the "for" loop requires 1 operation. Inside the loop, at line 3, there is 1 subtraction, 1 multiplication and 1 assignment operation, plus the 3 operations for updating the looping parameter, giving a total of 6 operations. The loop is executed $\mathtt{n} - 1$ times. Thus the number of basic operations required is

$$1 + 1 + 6(\mathtt{n} - 1) = 6\mathtt{n} - 4$$

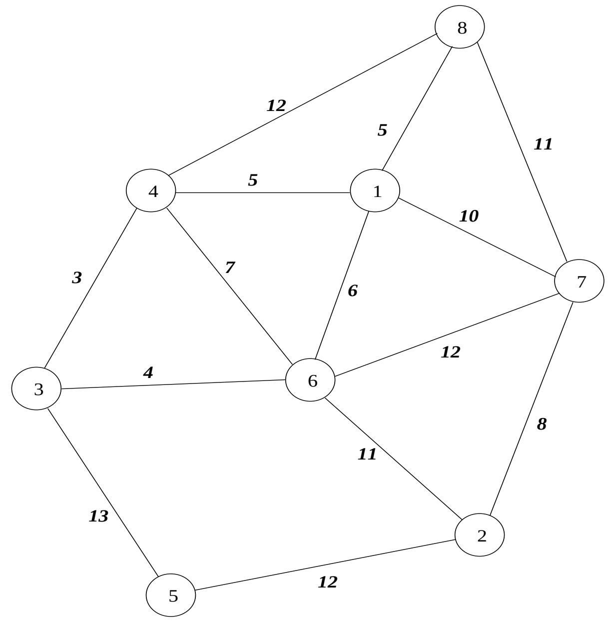
(b) Which best describes the computational complexity of this algorithm: $O(1)$, $O(n)$, $O(n \log n)$ or $O(n^2)$?

Solution: $O(n)$

Question 2

[17 marks]

Consider the undirected graph with 8 nodes and 14 arcs given below. Arc costs are indicated against each arc.

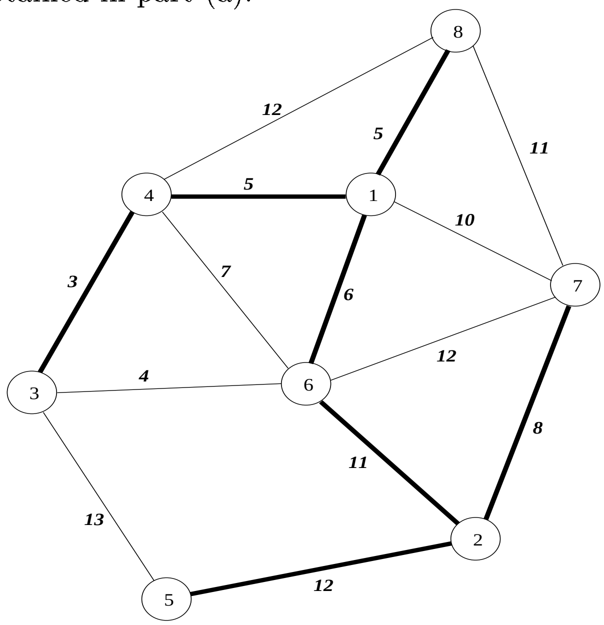


(a) Use Prim's algorithm to find the minimum cost spanning tree in the graph. Start the algorithm at node 1. Clearly indicate the order in which arcs/nodes were added to the tree, and the distance used by Prim's algorithm from each node to the tree at each step. You may do this by filling out the table below. The first line has already been done.

Solution:

	$(d(i), k(i))$							
i	1	2	3	4	5	6	7	8
	$(0,-)$	$(\infty,-)$	$(\infty,-)$	$(\infty,-)$	$(\infty,-)$	$(\infty,-)$	$(\infty,-)$	$(\infty,-)$
1		$(\infty,-)$	$(\infty,-)$	(5,1)	$(\infty,-)$	(6,1)	(10,1)	(5,1)
4		$(\infty,-)$	(3,4)		$(\infty,-)$	(6,1)	(10,1)	(5,1)
3		$(\infty,-)$			(13,3)	(4,3)	(10,1)	(5,1)
6		(11, 6)			(13,3)		(10,1)	(5,1)
8		(11, 6)			(13,3)		(10,1)	
7		(8,7)			(13,3)			
2					(12,2)			
5								

(b) Consider the spanning tree in the indicated by the bold lines in the figure below. You should observe that this is not a minimum cost spanning tree: its value is 50 which is higher than the value you should have obtained in part (a).



Explain briefly how this tree does not satisfy the path optimality condition for a minimum spanning tree. Be specific.

Solution: Arc $\{3,6\}$ not in the given tree induces the path $(3,4,1,6)$. Both arcs $\{4,1\}$ and $\{1,6\}$ in this path have higher cost (5 and 6 respectively) than arc $\{3,6\}$ (cost 4). This violates the path optimality condition, which states that every arc in the path induced by a non-tree arc must have cost no more than the cost of the arc inducing the path. Alternatively, arc $\{1,7\}$ not in the given tree induces the path $(1,6,2,7)$. Arc $\{6,2\}$ in this path has higher cost (11) than the cost of arc $\{1,7\}$ (10), also violating the path optimality condition.

Question 3
[27 marks]

Using the LP-based Branch-and-Bound method, solve the following Integer Program.

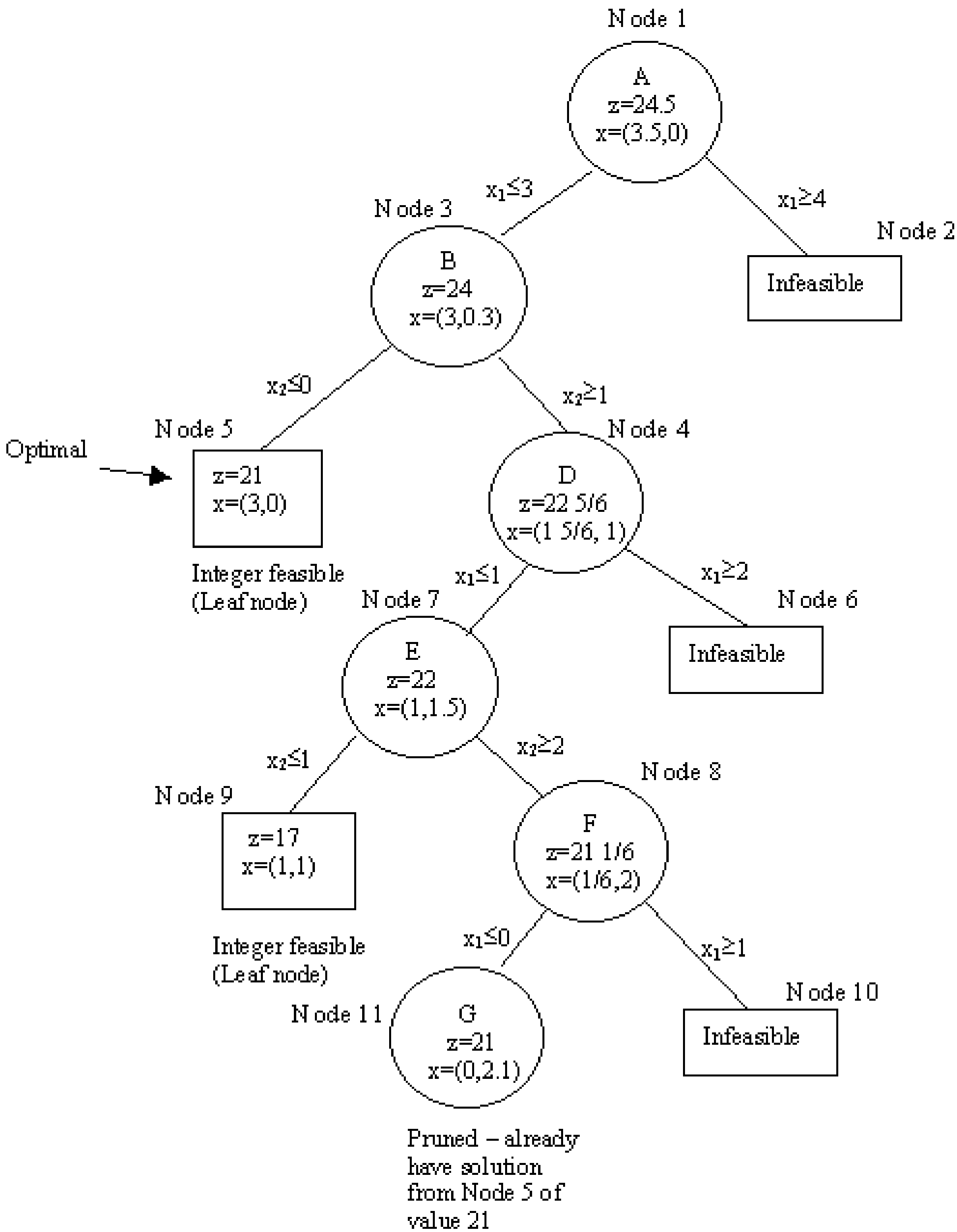
$$\begin{aligned} \max \quad z = \ & 7x_1 + 10x_2 \\ \text{s.t.} \quad & 6x_1 + 10x_2 \leq 21 \\ & x_1, x_2 \geq 0 \\ & x_1, x_2 \in \mathbb{Z} \end{aligned}$$

Clearly show the Branch-and-Bound tree, indicating the LP solution and LP value at each node for which the LP is feasible; at other nodes indicate clearly that the LP is infeasible. Number the nodes in the order explored. If you pruned the tree, explain clearly your reasons for doing so. You may find the graph below and some of the labelled points useful. All of the labelled points lie on the line $6x_1 + 10x_2 = 21$. Clearly state the optimal solution and indicate at which node of the Branch-and-Bound tree it was discovered. Hint: you may find it helpful to sketch the LP feasible region, which the points given below should assist. The solutions of the LPs which must be solved at each node of the Branch-and-Bound tree can be found by observation of your sketch.

Label	Point (x_1, x_2)	Objective Value z
A	(3.5, 0)	24.5
B	(3, 0.3)	24
C	(2, 0.9)	23
D	$(1\frac{5}{6}, 1)$	$22\frac{5}{6}$
E	(1, 1.5)	22
F	$(\frac{1}{6}, 2)$	$21\frac{1}{6}$
G	(0, 2.1)	21

Solution:

The branch-and-bound tree is given below. The optimal solution is $x = (3, 0)$ with optimal value $z = 21$, found at Node 5 of the branch-and-bound tree.



Question 4
[15 marks]

The following data represents the returns recorded by each of three assets over the last 6 months, as percentages, together with the expected return (mean) and variance of each asset, to two decimal places.

	Month							
Asset	1	2	3	4	5	6	Mean	Variance
1	12.0	7.5	4.5	9.0	-1.0	2.5	5.75	18.40
2	6.0	2.0	4.0	4.5	5.0	7.0	4.75	2.48
3	2.5	4.0	5.0	3.0	2.5	5.0	3.67	1.14

The covariance of the asset 1 with asset 2 return is -0.98, of the asset 1 with the asset 3 return is -1.17, and of the asset 2 with asset 3 return is -0.08, to two decimal places.

(a) Write down the covariance matrix for the returns of the three assets. Use Q to denote this matrix.

Solution:

$$Q = \begin{bmatrix} 18.4 & -0.98 & -1.17 \\ -0.98 & 2.48 & -0.08 \\ -1.17 & -0.08 & 1.14 \end{bmatrix}$$

(b) An investor with a budget of \$10,000 is considering investing her entire budget in some combination of these three assets, and wishes her portfolio to achieve an expected return of at least 4.6%. Write down a quadratic programming model of the problem of achieving her investment goals while minimizing her portfolio risk, where the risk is measured by variance of the portfolio return. Your model may be general or specific, however make sure you clearly define any parameters needed in your model, and describe clearly how they may be instantiated from the given data.

Solution: Define variables
x_i = amount invested in asset i, for $i = 1, \ldots, 3$.

The investors problem may be modelled as follows, where Q is the matrix defined in part (a).

$$\begin{aligned} \min \quad & x^T Q x \\ \text{s.t.} \quad & 5.75x_1 + 4.75x_2 + 3.67x_3 \geq 46000 \\ & x_1 + x_2 + x_3 = 10000 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

(c) In fact, a minimum investment of \$1,500 has been set for each asset, so either the portfolio does not include any of a particular asset, or the amount invested in the asset must be at least \$1,500. Show how to modify your model from part (b) to take account of this constraint, and so that the resulting mathematical constraints remain linear. Hint: you may need to introduce new integer variables.

Solution: Define new variables

$$y_i = \begin{cases} 1, & \text{if asset } i \text{ included in the portfolio} \\ 0, & \text{otherwise,} \end{cases} \qquad \text{for } i = 1, \ldots, 3.$$

We need to model the logical constraint that either $x_i = 0$ or $x_i \geq 1500$. We can do this by adding the constraints

$$x_i \leq 10000 y_i$$

and

$$x_i \geq 1500 y_i$$

for each $i = 1, \ldots, 3$.

Question 5
[12 marks]

Consider the following two piecewise linear functions of one variable:

$$c_1(x) = \begin{cases} \frac{2}{5}x, & 0 \leq x < 5 \\ x - 3, & 5 \leq x < 11 \\ 2x - 14, & 11 \leq x \leq 20. \end{cases}$$

and

$$c_2(x) = \begin{cases} 2x + 3, & 0 \leq x < 2 \\ 9 - x, & 2 \leq x < 4 \\ \frac{19-x}{3}, & 4 \leq x < 7 \\ \frac{4x-16}{3}, & 7 \leq x \leq 10. \end{cases}$$

Graphs of these functions are shown on the following page.

We are interested in solving the minimization problem:

$$\begin{aligned} \min \quad & c_1(x_1) + c_2(x_2) \\ \text{such that} \quad & 2x_1 - 3x_2 = 0 \\ & x_1 \in [0, 20] \\ & x_2 \in [0, 10]. \end{aligned}$$

Formulate the problem as a Linear Mixed Integer Programming problem. If either function can be modelled *without* using integer variables, do so. Do *not* attempt to solve the problem. Explain briefly how would you deduce a solution to the piecewise minimization problem above from the solution to your Linear Mixed Integer Programming problem.

Solution: Function c_1 is convex and so no integer variables are required. Function c_2 is neither convex nor concave, so binary variables will be needed.

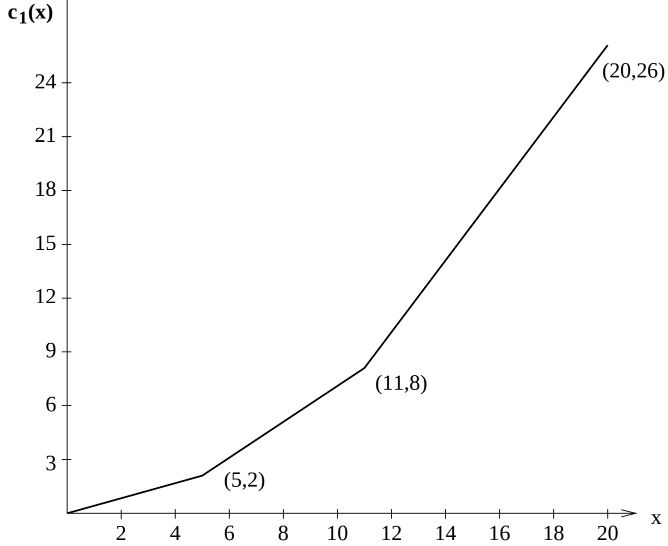
$$\begin{aligned}
\min \quad & \tfrac{2}{5}y_1 + y_2 + 2y_3 + 3l_1 + 7l_2 + 5l_3 + 4l_4 + 8l_5 \\
\text{s.t.} \quad & l_1 + l_2 + l_3 + l_4 + l_5 &= 1 \\
& z_1 + z_2 + z_3 + z_4 &= 1 \\
& l_1 &\le z_1 \\
& l_2 &\le z_1 + z_2 \\
& l_3 &\le z_2 + z_3 \\
& l_4 &\le z_3 + z_4 \\
& l_5 &\le z_4 \\
& 2y_1 + 2y_2 + 2y_3 - 6l_2 - 12l_3 - 21l_4 - 30l_5 &= 0 \\
& 0 \le y_1 \le 5 \\
& 0 \le y_2 \le 6 \\
& 0 \le y_3 \le 9 \\
& l_1, \ldots, l_5 \ge 0 \\
& z_1, \ldots, z_4 \in \{0, 1\}
\end{aligned}$$

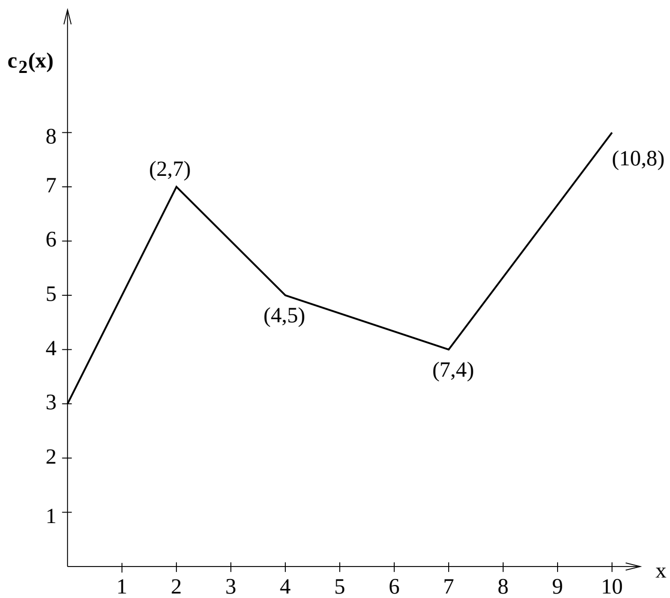
To deduce a solution to the piecewise minimization problem from a solution y, l, z of the MILP above, set

$$x_1 = y_1 + y_2$$

and

$$x_2 = 2l_2 + 4l_3 + 7l_4 + 10l_5.$$


c1(x)
24
21
18
15
12
9
6
3
(20,26)
(11,8)
(5,2)
2
4
6
8
10
12
14
16
18
20
x


c2(x)
8
7
6
5
4
3
2
1
(2,7)
(10,8)
(4,5)
(7,4)
1
2
3
4
5
6
7
8
9
10
x

Question 6
[7 marks]

An oil company is considering investing in up to n different projects. They must decide which projects will be funded, and which will not. Their Operations Research Division has set up a model, with a binary variable

$$x_i = \begin{cases} 1, & \text{if fund project } i \\ 0, & \text{otherwise} \end{cases}$$

for each $i = 1, \ldots, n$. Show how each of the following constraints would appear in the linear integer programming model.

(i) At most three of projects $1, 2, \ldots, 6$ can be funded.

Solution:

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \leq 3$$

(ii) If project 4 is not funded, then project 8 cannot be funded.

Solution:

$$x_8 \leq x_4$$

(iii) If both projects 5 and 7 are funded, then project 10 *must* be funded.

Solution:

$$x_5 + x_7 - 1 \leq x_{10}$$

Question 7
[15 marks]

A company called "Breezy" sells airconditioners in central and western Australia. The annual demand in each state is as follows: WA, 7,500; SA, 5,000; and NT, 6,000. Breezy is considering establishing distribution centres in each of Perth, Adelaide and Alice Springs. The cost of shipping an airconditioner from each of these cities to each region is given in the table below.

	WA	SA	NT
Perth	\$350	\$400	\$380
Adelaide	\$450	\$290	\$320
Alice Springs	\$390	\$300	\$280

Table 1: Cost per airconditioner of shipping from city to state

Any distribution centre can hold up to 7,000 aircondioners per year. The annual fixed cost of operating a distribution centre in each city is as follows: Perth, \$1 million; Adelaide, \$0.9 million; and Alice Springs, \$0.75 million.

Formulate the problem of deciding how to minimise the annual cost of supplying airconditioners as a Linear Integer Program. You may be as specific or general as you like, but if you take a general approach, be sure to clearly define all parameters of your model, and explain how they can be instantiated using the data given. Make sure you think about whether your model is as *good* as it could be. Do *not* attempt to solve the problem.

Solution:

Use $1, \ldots, 3$ to index potential distribution centres Perth, Adelaide and Alice Springs respectively. Use $1, \ldots, 3$ to index regions WA, SA and NT respectively.

Let c_{ij} denote the transportation cost from a distribution centre at i to region j, for each $i, j = 1, \ldots, 3$. So, for example, $c_{23} = 320$ and $c_{31} = 390$. Also let d_j denote the demand for region j, so for example, $d_1 = 7500$, since this is the demand for WA.

Variables:
x_{ij} = number of airconditioners shipped from distribution center i to region j per year,and

$$y_i = \begin{cases} 1, & \text{if distribution centre operating at } i, \\ 0, & \text{otherwise} \end{cases}$$

for all $i, j = 1, \ldots, 3$.

Now in order to ensure model is good, really should add

$$x_{ij} \leq \min\{7000, d_j\} y_i$$

for all $i, j = 1, \ldots, 3$. However this is implied for $j = 1$ by the distribution centre capacity constraints; it is only necessary to add these for $j = 2, 3$.

The model is as follows.

$$
\begin{aligned}
\min \quad & 1000000y_1 + 900000y_2 + 750000y_3 + \sum_{i=1}^{3}\sum_{j=1}^{3} c_{ij}x_{ij} & \\
\text{s.t.} \quad & \sum_{i=1}^{3} x_{ij} \geq d_j, & \forall j = 1, \ldots, 3 \\
& \sum_{j=1}^{3} x_{ij} \leq 7000y_i, & \forall i = 1, \ldots, 3 \\
& x_{i2} \leq 6000y_i, & \forall i = 1, \ldots, 3 \\
& x_{i3} \leq 5000y_i, & \forall i = 1, \ldots, 3 \\
& x_{ij} \geq 0 & \forall i, j = 1, \ldots, 3 \\
& y_i \in \{0, 1\}, & \forall i = 1, \ldots, 3
\end{aligned}
$$