(1) (a) (5 points) Let $A$ be the matrix that describes the projection of the plane onto a line $l$ through the origin. Explain from a geometric point of view what you expect $A^2$ to be and why.

(b) (5 points) Write down the formula for the projection of $x$ onto the line spanned by $\begin{pmatrix} a \\ b \end{pmatrix}$.

(c) (5 points) Write down the matrix $A$ describing the projection onto the line spanned by $\begin{pmatrix} a \\ b \end{pmatrix}$.

(d) (5 points) Compute $A^2$, and compare the result with your answer to (a).

(e) (5 points) From a geometric point of view, what do you think is $\det(A)$ and why?

(f) (5 points) Compute $\det(A)$ to check your answer to the previous question.

(g) (5 points) Is $A$ an orthogonal matrix?

(h) (5 points) What is the orthogonal projection of $\begin{pmatrix} 8 \\ 2 \end{pmatrix}$ to the line spanned by $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$?

(i) (5 points) What is the orthogonal projection of $\begin{pmatrix} 8 \\ 2 \end{pmatrix}$ to the line spanned by $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$?

Date: March 27, 2007.
(j) (5 points) Without further calculation, express \( \begin{pmatrix} 8 \\ 2 \end{pmatrix} \) as a linear combination of \( \begin{pmatrix} 2 \\ 1 \end{pmatrix} \) and \( \begin{pmatrix} 1 \\ -2 \end{pmatrix} \), and explain why your answer works. [Hint: if you don’t know what to do and want to cheat a little, you can calculate what the linear combination should be and then interpret it using the answers to the previous questions.]

(2) (a) (5 points) Let \( A \) be the matrix that describes the reflection of the plane at a line \( l \) through the origin. Explain from a geometric point of view what you expect \( A^2 \) to be and why.

(b) (5 points) Write down the formula for the reflection of \( x \) at the line spanned by \( \begin{pmatrix} a \\ b \end{pmatrix} \).

(c) (5 points) Write down the matrix \( A \) describing the reflection at the line spanned by \( \begin{pmatrix} a \\ b \end{pmatrix} \).

(d) (5 points) Compute \( A^2 \), and compare the result with your answer to (a).

(e) (5 points) Is \( A \) an orthogonal matrix?

(f) (5 points) From a geometric point of view, what do you think is \( \det(A) \) and why?

(g) (5 points) Compute \( \det(A) \) to check your answer to the previous question.

(h) (5 points) What is the reflection of \( \begin{pmatrix} 1 \\ 1 \end{pmatrix} \) at the line spanned by \( \begin{pmatrix} 2 \\ -4 \end{pmatrix} \)?

(3) Fourier series: We are going to work with the vector space \( L^2([-\pi, \pi]) \) of integrable complex valued functions on the interval \([-\pi, \pi]\). [In case you don’t know the complex numbers,
you may take your functions to be real valued.] On this vector space, we define the following inner product:

\[ \langle f, g \rangle := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \cdot g(x) \, dx. \]

Or, if you prefer working over the real numbers,

\[ \langle f, g \rangle := \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cdot g(x) \, dx. \]

(a) (10 points) Show that \( \langle -,- \rangle \) is indeed an inner product. Hint: to show this, you need to show that it satisfies:

(i) \( \langle f_1 + f_2, g \rangle = \langle f_1, g \rangle + \langle f_2, g \rangle \)

(ii) \( \langle \lambda f, g \rangle = \lambda \langle f, g \rangle \), where \( \lambda \) is a (complex) scalar.

(iii) \( \langle f_1, g_1 + g_2 \rangle = \langle f_1, g_1 \rangle + \langle f_1, g_2 \rangle \)

(iv) \( \langle f, \lambda g \rangle = \overline{\lambda} \langle f, g \rangle \), where \( \lambda \) is a complex scalar. [If \( \lambda \) is real, then \( \overline{\lambda} = \lambda \).]

(b) (optional) Show that the functions

\[ f_n(x) = e^{ix} \]

with \( n \in \mathbb{Z} \) form an orthonormal system. If you are working over the reals, you need to consider two kinds of functions:

\[ f_n(x) = \cos(nx) \]

for \( n \geq 0 \) and

\[ g_n(x) = \sin(nx) \]

for \( n \geq 1 \).

[Hint: to show this, you have to check the following conditions:

(i) For every \( n \in \mathbb{Z} \), we need \( \langle f_n, f_n \rangle = 1 \), (and, if you are working over the reals, the same for \( g_n \)),

(ii) For \( n \neq m \in \mathbb{Z} \), we need \( \langle f_n, f_m \rangle = 0 \), (and, if you are working over the reals, the same for \( g_n \) and \( g_m \)), and

(iii) For every \( n, m \in \mathbb{N} \), we need \( \langle f_n, g_m \rangle = 0 \) (you only need to do this part if you are working over the reals.)]
(c) (optional) What is the orthogonal projection of

\[ h(x) = \pi^2 - x^2 \]

to \( \sin(nx) \)? Using integration by parts, calculate the orthogonal projection of \( h(x) \) to \( \cos(nx) \).

Hint: you can find the solution at

http://webpages.dcu.ie/~applebyj/ms224/BT2__FSER.pdf