Look up the definitions of the following objects in the book: function (also called “map”), domain (also called “source”), target, image, graph, well-defined, injective (also called “one-to-one”), surjective (also called “onto”), bijective (also called “one-to-one correspondence”).

(1) Consider the map

\[ f: \mathbb{N} \rightarrow \mathbb{N} \]
\[ x \mapsto 2x. \]

(Here we used the notation “\( x \mapsto 2x \)” for “\( f(x) = 2x \).”)

(a) What is the image of \( f \)?
(b) Is \( f \) surjective?
(c) Is \( f \) injective?

(2) Let \( X \) and \( Y \) be the sets

\[ X = \{a, b, c, d\} \]
\[ Y = \{a, b, e\} \]

Draw the graph of each of the following, decide whether it is a well-defined map and if so, whether it is injective, surjective or bijective: (It it possible that several or none of these hold.)

\[ f: X \rightarrow Y \]
\[ a \mapsto b \]
\[ b \mapsto c \]
\[ c \mapsto a \]
\[ d \mapsto b \]

\[ g: X \rightarrow Y \]
\[ b \mapsto e \]
\[ c \mapsto a \]
\[ d \mapsto b \]

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h: Y → X
  a ↦ b
  b ↦ a
  e ↦ a

i: Y → X
  a ↦ b
  b ↦ c
  a ↦ a
  e ↦ d

j: Y → X
  a ↦ b
  b ↦ c
  e ↦ d

Try to formulate for a general map $f$ what the properties “well-defined”, “injective”, “surjective” or “bijective” mean for the graph.

(3) Let now $X$ and $Y$ be arbitrary sets, and let $f: X \to Y$ and $g: Y \to X$ be two maps. The map $g$ is called a right inverse to $f$, if (and only if) for every element $y$ of $Y$ the equality

$$y = f(g(y))$$

holds. The map $g$ is called a left inverse to $f$, if (and only if) for every element $x$ of $X$ the equality

$$x = g(f(x))$$

holds. The map $g$ is called an inverse of $f$, if (and only if) it is both a right and left inverse of $f$.

Prove the following statements:
(a) A map $f$ is surjective if and only if there exists a right inverse of $f$.
(b) A map $f$ is injective if and only if there exists a left inverse for $f$.
(c) A map $f$ is bijective if and only if there exists an inverse of $f$. 