(1) (35 points) Do Problem (1) of last year’s take-home final.

(2) (15 points) We consider the function (= map)

\[ f: \mathbb{R} \to \mathbb{R} \]
\[ f(x) := x^2. \]

Prove (formally, not with calculus-type arguments) the following statement:

For every real number \( x \) and every real number \( \epsilon > 0 \), there exists a real number \( \delta > 0 \) such that for any real number \( y \) with \( |x - y| < \delta \), we have \( |f(x) - f(y)| < \epsilon \).

(3) Fix a finite set \( X \). We define the set \( \mathcal{S}(X) \) as follows:

\[ \mathcal{S}(X) := \{ Y \mid Y \subseteq X \}. \]

(So the elements of the set \( \mathcal{S}(X) \) are themselves sets, and a set \( Y \) is an element of the set \( \mathcal{S}(X) \) if and only if \( Y \) is a subset of \( X \)).

(a) (4 points): List all the elements of \( \mathcal{S}(\emptyset) \), of \( \mathcal{S}\{x\} \), of \( \mathcal{S}\{x,y\} \) and of \( \mathcal{S}\{x,y,z\} \). Make an educated guess how to fill in the blank in the following sentence:
Let $n$ be the number of elements of $X$. Then the number of elements of $S(X)$ equals . . .

(b) **(13 points)** Use the principle of induction to prove your guess from part (a).

(4) **(14 points):** Fix a natural number $m$, and consider the equivalence relation of the example in Problem (2) of Problem Set 5 with 3 replaced by $m$. Let $Y$ be the set of all equivalence classes of this equivalence relation. I.e.,

$$Y := \{[x] \mid x \in \mathbb{Z}\} = \{[0], [1], \ldots, [m-1]\}.$$  

Careful, the elements of $Y$ are sets, which in turn contain integers. We define the operation $*$ as follows: for two classes $[x]$ and $[y]$ in $Y$, the class $[x] * [y]$ is given by

$$[x] * [y] := [x + y].$$

You have proved in Problem Set 5, Problem (2d) that $*$ is well defined (at least in the case that $m = 3$, but one could prove it for arbitrary $m$ in exactly the same way).

Give complete proofs for the following statements:

(a) There exists a class $[x_0] \in Y$ such that for every class $[a] \in Y$, one has $[a] * [x_0] = [a]$.

(b) For any class $[b] \in Y$ there exists a class $[c] \in Y$ such that $[b] * [c] = [x_0]$.

(5) What is the largest real number $r$ such that, whenever $x$ is a real number with absolute value $|x| < r$, the inequality

$$x^2 \leq 3|x|$$

holds?

(a) **(3 points)** Find $r$.

(b) **(7 points)** Write down a formal proof that, with your $r$ from part (a), for every real number $x$ whose absolute value $|x|$ is strictly less then $r$ the equation

$$x^2 \leq 3|x|$$

holds.

(c) **(9 points)** Prove that your $r$ is the largest real number with this property.