§ 1: The basic moduli problem

Let $E_\ast$ be a homotopy theory based on a ring spectrum $E$, so that $\eta: E_\ast \to E_\ast E$ is flat (\Rightarrow good theory of Vectorial).

Let $A$ be a commutative algebra in $E_\ast E$-comodule ("comodule algebra"). Find all $E_\infty$ ring spectra $X$ so that $E_\ast X \cong A$.

Category $\text{RCA}$ "realization category".

$\text{Ob } \text{RCA} = \{X \in \text{Alg}_{E_\infty} \mid \text{so that } E_\ast X \cong A \text{ (not part of data)}\}$

$\text{mor } \text{RCA} = \{X \to Y \in \text{Alg}_{E_\infty} \mid \text{so that } E_\ast f \text{ is iso} \}$

Moduli space for this problem: $\text{BRCA} = \text{ nerve of } \text{RCA}$.

Basic problem: Calculate the homotopy type of $\text{BRCA}$ if $\text{BRCA} \neq \emptyset$.

If $\text{BRCA} \neq \emptyset$ $\Rightarrow \exists X$ so that $E_\ast X \cong A$

Then: $\pi_0 \text{BRCA} = \text{E}_{\infty}$-coarse homotopy of $X$'s.

(Thm (Dwyer, Kan))

$\text{BRCA} \cong \coprod_{[X] \in \pi_0} \text{hom}_{E_\infty}(X)$

with $\text{hom}_{E_\infty}(X)$ space of self $E_\ast$-equivalence of some fibrant/cofibrant $X$ in $\text{E}_{\infty}$.

§ 2: Spectra and $E_\infty$ ring spectra

$\Sigma$ = a category of spectra

Axiom 1: $\text{Ho} (\Sigma) = \text{stable homotopy category}$

$\Sigma$ is a cofibrantly generated monoidal model category Quillen equivalent to Boardman-Vogt's operad spectra. Technical: generators should have cofibrant source.
Axiom 2: $S$ has closed symmetric monoidal smash product which descends to the usual smash product on homotopy category.

Axiom 3: If $S$ is the sphere spectrum and $X$ a space, then

$$K \otimes X \cong (K \otimes S) \otimes X \cong (\text{i.e., } 2 \otimes X\text{ is cupable}).$$

Axiom 4: If $K$ is a multiplicative $\mathbb{Z}_n$-space, which is a weak equivalence to spaces, and $X$ is a cofibrant spectrum, then

$$K \otimes (X^{\otimes \infty}) \cong L \otimes (X^{\otimes \infty})$$

is a weak equivalence.

Theorem: Such $S$ exist.

Examples:
- $S$-modules
- $HSS$ symmetric spectra on top spaces in pointed model structure
- $HSS$ symmetric spectra on simplicial sets

Example: If $E \mathbb{Z}_n$ is your favorite free contractible $\mathbb{Z}_n$-space, then $E \mathbb{Z}_n \to X$ satisfies hypothesis of Axiom 4, so

$$E \mathbb{Z}_n \otimes X^{\otimes \infty} \to X^{\otimes \infty} / E \mathbb{Z}_n$$

is a weak equivalence for cofibrant $X$.

Definition: $E\infty$-rings are $\infty$ categories of commutative monoids in $S$ under $\otimes$.

$$= \text{Alg}_{E\infty}.$$

Axiom 5: For any opd of simplicial sets, the category $\text{Alg}_{E\infty}$ inherits a model category structure (i.e., fibrations and weak equivalences defined in $S$).
Andre-Quillen cohomology.

Suppose $X \to Y$ is a morphism of $E_0$-modules. Want to calculate

$$\pi_t \text{ map}_{E_0}(X, Y) \cong \text{map}_{E_0}(X, Y_0) \Rightarrow \text{Hom}_{E_0}(E_0X, E_0Y)

\text{Hom}_{E_0}(Y^0) = E_0Y \oplus E_0E_0Y \cong E_0Y [E_0E, E_0, E_0E_0]

\text{Andre-Quillen cohomology is the derived functor of this.}

If $A$ is a $k$-algebra, $M$ an $A$-module, the "square $2 \to 0$" extending

$A \otimes M = A \otimes M$ with product

$$K(A, n) = K(A, n) \cong (a, x)(b, y) = (ab, xb + ay)

\text{Then have simplicial descent $A$-module whose normalized is } H^0 \otimes\text{J}

A \otimes K(M, n) \in \text{Alg}_k

\text{Then } H^n(A/k, M) = \text{map}_{\text{Alg}_k / A}(A, A \otimes K(M, n))

\text{Here and in following all spaces of maps are derived, replace source and target by coherent resp. projective objects as necessary.}

\text{Then } H^n(A/k, M) = \pi_0 H^n(A, M) = \pi_k H^0 \otimes\text{J}(A, M).

\text{Note: } E_0(X^0) = E_0Y \otimes(E_0E_0Y) \text{ is an object of } E_0E \text{- module algebra.}

\text{If } A \text{ is } E_0 \text{- module object and } M \text{ an } A \text{- module in } E_0 \text{- modules, then define similarly}

\text{If } M = E_0E \otimes\text{N}_0 \text{ is an extended comodule, then}

\text{If } M = E_0E \otimes\text{N}_0 \text{ is an extended comodule, then}

$H^n_{E_0E}(A/E_0, E_0E \otimes\text{N}_0) \cong H^n(A/E_0, M)$.
Remark: \( \mathcal{X} (A, M) \) depends very much on the graded category, which is supposed from notation:
- associative algebras
- commutative algebra (does not apply, since \( E_0 \) (Grothendieck) has extra structure)

One needs that \( E_0 (\text{free } E_0 (x)) = \text{some functor of } E_0 X \).

Note: for decent \( E_0 \), \( E_0 (\text{Tensur } x) = \text{Tensur } (E_0 X) \)
if \( E_0 X \) is flat as \( E_0 \)-module.

Consider \( \text{Alg}_0 \) with \( E_0 \)-iso morphisms as weak equivalences.
(localization of Ankin S model structure).

Thm: Let \( f : X \rightarrow Y \) be a morphism in \( \text{Alg}_0 \). Then there is a spectral sequence
\[
E_2^{s,t} = \Rightarrow \pi_{t-s} \left( (\text{Alg}_0, (X, Y), f) \right)
\]
with \( E_2^{0,0} = \text{Hom}_{E_0 \text{-alg}} (E_0 X, E_0 Y) \) and
\[
E_2^{s,t} = H^{s}_{\text{assoc}} \left( \frac{E_0 X}{E_0 Y}, \mathbb{Z}^{|E_0 Y|} \right)
\]
where \( Y_{\mathbb{E}} = \mathbb{E} \)-completion of \( Y \).

Bun's field: given algebraic map \( f : E_0 X \rightarrow E_0 Y \), there are obstructions
\[
\Theta_f \in H^{s+1}_{\text{assoc}} \left( \frac{E_0 X}{E_0 Y}, \mathbb{Z}^{|E_0 Y|} \right)
\]
so realizing \( f \) as a map of \( \text{Alg}_0 \)-algebras \( X \rightarrow Y \).

(Obstructions lie in "(1)-stem").
**Theorem:** There exist successively defined obstructions

\[ s_s \in H_{s+1}^{\text{assoc}}(\text{E}_{s+1}, \text{E}_s \times \text{C}_{s+1}) \]

to realizing \( \text{A}_{s} \) as an \( \text{A}_s \)-algebra

(Here \( (\text{M}_{1}\text{E}_{1})_n = \text{M}_{1,n} \), so that \( \text{M}_{1}\text{E}_{1} = \) \( \mathbb{Z} \times \text{M} \))

Having such a realization gives a spectral sequence

\[ H_{s+1}^{\text{E}_{s+1}-\text{assoc}}(\text{E}_{s+1}, \text{E}_s \times \text{C}_{s+1}) \Rightarrow \pi_{s+1}^{\text{E}_{s+1}} \text{BR}(\text{A}_s) \]

14. 10. 03

---

**Recall:** We defined \( \text{A}_s \)-cohomology objects

\[ H_{s+1}^{\text{assoc}}(\text{A}, \text{K}; \text{M}) = \pi_0 \text{map}_{k-\text{alg}/\text{A}}(\text{A}, \text{A} \times \text{K}(\text{M}, \text{m})) \]

\[ \text{ob} \left( k-\text{alg}/\text{A} \right) = \left\{ \begin{array}{cl} \text{B} & \text{for } n \neq 0 \\ \text{A} & \text{for } n = 0 \end{array} \right. \]

This was a respective \( \text{E}_s \text{E}_s \)-comodule version; and the \( \text{A}_s \)-cohomology objects came up in obstructions theory.

---

**\( \text{A}_s \)-cohomology for \( \text{E}_s \text{E}_s \)-algebras**

To form the derived space of maps, need to take a cofibrant replacement \( X_0 \to \text{A} \) in simplicial associative algebras. Thus

\[ \pi_n X_0 = \left\{ \begin{array}{cl} 0 & \text{for } n \neq 0 \\ \text{A} & \text{for } n = 0 \end{array} \right. \]

Cofibrant: forgetting the face maps, \( X_0 \simeq \text{Tensor}_k(\text{M}) \) for some simplicial \( k \)-module which is projective in each dimension.

If \( X \in \text{Alg}_{0,0} \), we can imagine a simplicial \( \text{A}_0 \)-ring spectrum \( Z \to Y \) such that, forgetting the face maps, \( Z_0 \simeq \text{Tensor}_k(\text{M}_0) \) such that \( E_0(\text{M}_0) \) is a projective \( \text{E}_1 \)-module.
Then \( E_\ast \otimes E_\ast \otimes \Gamma(\Pi) \cong \text{Tensor}_{E_\ast} (E_\ast \otimes \Pi) \).

and \( \eta_\ast (E_\ast Y_\ast) \Rightarrow E_\ast X \).

Crucial observation

\( \text{Spec}_k = \text{Simplicial } k\text{-algebras} = \text{algebras in \textbf{Simplicial} } k\text{-modules.} \)

I.e. \( A \in \text{Spec}_k \) is equipped with \( A \otimes_k A \rightarrow A \).

Let \( \mathcal{C} \) be any category with colimits and \( K \) a simplicial set. Then for \( X_\ast \in \mathcal{C} \)

\( \text{K}\otimes X_\ast \in \mathcal{C} \) given by \( (K \otimes X)_n = \underleftarrow{\lim} X_n \).

Warning: If \( X_\ast \in \mathcal{S} \) is a simplicial spectrum, then

\( K \otimes X_\ast \neq \prod X_n \).

Example: If \( M \in \text{Spec}_k \), then \( K \otimes M = \text{K} \text{End}_k \).

If \( X_\ast \in \text{Spec} \) is simplicial spectrum, then

\( [K \otimes X]^n = (Kn)_+ \otimes X_n \).

If \( \mathcal{F} \) is any operad in simplicial sets and \( \mathcal{C} \) has a symmetric monoidal \( \wedge \)
structure, then we have \( \mathcal{F}\text{-algebras} \) in \( \mathcal{S} \), i.e. \( X_\ast \in \mathcal{S} \)

\( \mathcal{F}(n) \otimes X_n \leftarrow X_n \rightarrow X. \)

In particular, in each simplicial dimension \( k \), \( \mathcal{F}(k) \) is a set-operad and \( X_k \) is an \( \mathcal{F}(k) \)-algebra.
Example: Let $s(n) = \text{Ass}(n) = \mathbb{Z}_n$, an abelian simplicial set then $s$-algebras in $s$-modules are precisely $s$-algebras.

**Definition:** An $E_\infty$-operad $E$ is a simplicial set operad such that
1. for all $n, k$, $E(n)_k$ is a free $\mathbb{Z}_n$-set and $E(n)_k$ is weakly contracting.
2. $E(sE) = \text{algebra over the operad } E$ in $sE$.

**Proposition:**
1. Let $X$ be a simplicial spectrum and $E$ an $E_\infty$-operad in simplicial sets. Then if $E*$ has a K"{a}hler spectral sequence, and $E*_kX_n$ is projective for each $n$, then
   $$E_*(E(X_*)) \cong E_*(E*_kX_*)$$
   Here $E(-)$ is free algebra in $sE$, with $E$ understood.
2. If $X_* \in E(sE)$, then the geometric realization $|X_*|$ is a spectrum in an $E_\infty$-operad spectrum.

**Proposition:** The category $E(sE)$, $E(s\text{Mod}_E)$ are in dependent up to chain equivalence are independent of $E$ (with $\pi_0E_*(-)$ equalizers resp. $\pi_0$-equalizers).

**Proof of upper Prop. Part 1:**
$$[E_*(E(X_*))]_k \cong E_*(\bigvee_{n \geq 0} (E(n)_k)_+ \wedge X_{k+1} \wedge X_k) \cong \bigoplus_{n \geq 0} E_*(E(n)_k) \otimes (E_*X_k) \otimes E_*(\mathbb{Z}_n) \cong E_*(E_*X_k)$$
Example: If \( A \) is a simplicial commutative \( k \)-algebra, then

\[
E \to \mathbb{C}om\text{on}
\]

make \( A \) into an object in \( E(\text{Mod}_k) \).

In particular, this goes for the content simplicial algebra.
If \( M \) is an \( A \)-module (for \( A \) discrete/contient), then

\[
B_0 = A \times_k (M, n) \in E(\text{Mod}_k).
\]

\( \Delta^n \): AQ cohomology:

\[
H^n_\mathbb{A}(\mathbb{A}_1^I, M) = \Delta^n \to \text{map} \left( E(\text{Mod}_k) / A, A \times_k (M, n) \right)
\]

\( \Delta^n (A, M) \): derived mapping space.

Robinson-Whitehouse: connecting theory using \( \mathbb{A} \)-homology.

Theorem: If \( A \) is a commutative algebra in \( E \)-\( E \)-comodules

under certain hypotheses in \( E \), there are uniquely defined obstruction

\[
E_2 = H^{s+2}_{E-E} (A, A \otimes_{\text{A}} S \text{J})
\]

in realizing \( A \) as an \( E \)-\( E \) ring spectrum.

There is a spectral sequence for analyzing the entire moduli space

of \( E \)-realizations of \( A \).

Proposition: \( H^* (A/k, M) \) satisfies flat base change,

compatibility, and vanishes when \( k \to A \) is etale.

Flat base change:

\[
L \to L \quad \text{If } f \circ g \text{ is flat, and } M \text{ is an } A_{\text{et}} - \text{module. Then }
\]

\[
H^* (A/k, M) \cong H^* (L \to A, A; M)
\]