THE DISTRIBUTION OF THE FIRST EIGENVALUE SPACING AT THE HARD EDGE OF THE LAGUERRE UNITARY ENSEMBLE

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Abstract. The distribution function for the first eigenvalue spacing in the Laguerre unitary ensemble of finite rank random matrices is found in terms of a Painlevé V system, and the solution of its associated linear isomonodromic system. In particular it is characterised by the polynomial solutions to the isomonodromic equations which are also orthogonal with respect to a deformation of the Laguerre weight. In the scaling to the hard edge regime we find an analogous situation where a certain Painlevé III’ system and its associated linear isomonodromic system characterise the scaled distribution. We undertake extensive analytical studies of this system and use this knowledge to accurately compute the distribution and its moments for various values of the parameter $a$. In particular choosing $a = \pm 1/2$ allows the first eigenvalue spacing distribution for random real orthogonal matrices to be computed.

1. Introduction

The Laguerre unitary ensemble (LUE$_{n,a}$) of random matrices is specified by the eigenvalue probability density function (p.d.f.)

\begin{equation}
(1.1) \quad p(\lambda_1, \ldots, \lambda_n) := \frac{1}{n!c_{n,n+a}} \prod_{j=1}^{n} e^{-\lambda_j} \lambda_j^a \prod_{1 \leq j < k \leq n} (\lambda_k - \lambda_j)^2, \quad \lambda_1, \ldots, \lambda_n \in [0, \infty),
\end{equation}

where

\begin{equation}
(1.2) \quad c_{m,n} := \frac{1}{m!} \prod_{j=1}^{m} \Gamma(n - j + 1) \Gamma(m - j + 2).
\end{equation}

The naming relates to the fact that (1.1) is the eigenvalue p.d.f. of complex Hermitian matrices $X$ with measure invariant under unitary conjugation $X \mapsto UXU^{-1}$, proportional to the generalised Laguerre form

\begin{equation}
(1.3) \quad (\det X)^a e^{-\text{Tr}(X)}.
\end{equation}

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In multivariate statistics (1.1) is realised as the eigenvalue p.d.f. for the complex case of the so-called Wishart matrices $X = Y^\dagger Y$. Here $Y$ is an $N \times n$ ($N \geq n$) rectangular matrix of i.i.d. entries with distribution $N[0,1] + iN[0,1]$. In this setting $a = N - n$, and so $a$ is naturally a non-negative integer. The spectrum of complex Wishart matrices has found recent application in studies of wireless communication systems [34], where the matrix $Y$ consists of the complex amplitudes of various channels of transmitted waves as received by the antennas.

The matrix structure $Y^\dagger Y$ is relevant to the study of the eigenvalues of the $(n + N) \times (n + N)$ Hermitian matrix

$$\tilde{X} := \begin{pmatrix} 0_{N \times N} & Y \\ Y^\dagger & 0_{n \times n} \end{pmatrix}.\quad (1.4)$$

Thus one has that $\tilde{X}$ has in general $N - n$ zero eigenvalues, with the remaining eigenvalues given by $\pm$ the positive square roots of the eigenvalues of $Y^\dagger Y$ (see e.g. [9]). This matrix structure has application to the study of the Dirac equation in the context of quantum chromodynamics [36]. There most interest is in the scaling behaviour of the smallest eigenvalues.

In the study of matrix Lie algebras one encounters antisymmetric matrices $(X^T = -X)$ with pure imaginary complex elements. Specifically, such matrices are the Hermitian part of the matrix Lie algebra

$$i \times (so(n, \mathbb{C})) := \{ i \text{ times } n \times n \text{ skew symmetric complex matrices} \}.$$ \quad (1.5)

If the independent imaginary complex elements are i.i.d with distribution $iN[0,1]$, then the p.d.f. of the positive eigenvalues is proportional to

$$n = 2m \text{ even: } \prod_{j=1}^{m} \exp(-\lambda_j^2) \prod_{1 \leq j < k \leq m} (\lambda_k^2 - \lambda_j^2)^2, \quad (1.6)$$

$$n = 2m + 1 \text{ odd: } \prod_{j=1}^{m} \lambda_j^2 \exp(-\lambda_j^2) \prod_{1 \leq j < k \leq m} (\lambda_k^2 - \lambda_j^2)^2. \quad (1.7)$$

This ensemble will be denoted by $AS(n)$. Under the change of variables $\lambda_j^2 \mapsto \lambda_j$ these reduce to the LUE with parameters $a = -1/2, 1/2$ respectively.

Antisymmetric Hermitian matrices $X$ can be used to parameterise real orthogonal matrices $R$ with determinant $+1$ (and thus, by definition, members of the classical group $O^+(n)$). Thus we can write $R$ according to a Cayley transformation

$$R = \frac{I_n + iX}{I_n - iX}. \quad (1.8)$$

Note from this that the property that the eigenvalues of $X$ come in $\pm$ pairs is consistent with the property that the eigenvalues of $R$ come in complex conjugate pairs $e^{\pm i\theta}$. This can be used (see e.g. [9]) to show that with the matrix $R$ chosen
with uniform (Haar) measure, the eigenvalue p.d.f for the eigenvalues with angles $0 \leq \theta \leq \pi$ is proportional to

\begin{equation}
(1.9) \quad n = 2m \text{ even: } \prod_{1 \leq j < k \leq m} (\cos \theta_k - \cos \theta_j)^2,
\end{equation}

\begin{equation}
(1.10) \quad n = 2m + 1 \text{ odd: } \prod_{j=1}^{m} (1 - \cos \theta_j) \prod_{1 \leq j < k \leq m} (\cos \theta_k - \cos \theta_j)^2.
\end{equation}

Note that for $\theta \to 0$ these have the same leading behaviour as (1.6), (1.7) with $\lambda \to 0$. This is consistent with the fact that the $m \to \infty$ scaled joint distribution function for the $k$ smallest eigenvalues, $p_{(k)}$ say, is the same for both ensembles, (1.11)

\begin{equation}
\lim_{m \to \infty} \left( \frac{\pi}{2 \sqrt{m}} \right)^k p_{AS(n)}^{\text{LUE}} \left( \frac{\pi X_1}{2 \sqrt{m}}, \ldots, \frac{\pi X_k}{2 \sqrt{m}} \right) = \lim_{m \to \infty} \left( \frac{\pi}{m} \right)^k p_{(k)}^{O^+(n)} \left( \frac{\pi X_1}{m}, \ldots, \frac{\pi X_k}{m} \right),
\end{equation}

where $n = 2m, 2m + 1$. This scaling is such that the average spacing between eigenvalues approaches unity as $k \to \infty$. The distribution (1.11) is of primary importance in the study of the spectral interpretation of $L$-functions [20],[30]. From the remark below (1.7) we know that

\begin{equation}
(1.12) \quad \left( \frac{1}{2 \sqrt{x_1}} \right) \cdots \left( \frac{1}{2 \sqrt{x_k}} \right) p_{AS(n)}^{\text{LUE}} (\sqrt{x_1}, \ldots, \sqrt{x_k}) = p_{LUE_{m,a}}^{(1)} (x_1, \ldots, x_k),
\end{equation}

where for $n = 2m$, $a = -1/2$, while for $n = 2m + 1$, $a = 1/2$. Consequently

\begin{equation}
(1.13) \quad \lim_{m \to \infty} \left( \frac{\pi}{m} \right)^k p_{(k)}^{O^+(n)} \left( \frac{\pi X_1}{m}, \ldots, \frac{\pi X_k}{m} \right) = \lim_{m \to \infty} \left( \frac{\pi^2}{2m} \right)^k X_1 \cdots X_k p_{LUE_{m,a}}^{(1)} \left( \frac{\pi^2 X_1^2}{4m}, \ldots, \frac{\pi^2 X_k^2}{4m} \right).
\end{equation}

Thus knowledge of the distribution $p_{LUE_{m,a}}^{(1)}$ for $a = \pm 1/2$ suffices to compute the scaled limit of $p_{(k)}^{O^+(n)}$.

In the case $k = 1$ a number of different characterisations of $p_{LUE_{m,a}}^{(1)}$, which is the distribution of the smallest eigenvalue in $LUE_{m,a}$, are known. First, with $n \mapsto n + 1$ in (1.1) for convenience, the p.d.f. of the smallest eigenvalue is given by fixing one of the coordinates at $x_1$, and integrating the remaining over $[x_1, \infty)$. Thus

\begin{equation}
(1.14) \quad p_{LUE_{m+1,a}}^{(1)} (x_1) = \frac{1}{n! c_{n+1,n+1+a}} e^{-x_1 x_1^a} \times \int_{x_1}^{\infty} d \lambda_1 \ldots \int_{x_1}^{\infty} d \lambda_n \prod_{j=1}^{n} e^{-\lambda_j} \lambda_j^a (\lambda_j - x_1)^2 \prod_{1 \leq j < k \leq n} (\lambda_k - \lambda_j)^2.
\end{equation}

One has that

\begin{equation}
(1.15) \quad p_{LUE_{m+1,a}}^{(1)} (x_1) = -\frac{d}{dx_1} E^{LUE_{m+1,a}} (x_1),
\end{equation}
where
\begin{equation}
E_{\text{LUE}}^{n+1,a}(t) := \int_t^\infty d\lambda_1 \cdots \int_t^\infty d\lambda_{n+1} p(\lambda_1, \ldots, \lambda_{n+1}),
\end{equation}
is the probability (gap probability) that no eigenvalues are in the interval \((0, t)\).

For integer values of the parameter \(a\) in (1.1), \(E_{\text{LUE}}^{n,a}(t)\) was studied by orthogonal polynomial techniques in [11], where it was evaluated as an \(a \times a\) determinant, and by the method of Jack polynomials in [10], giving an \(a\)-dimensional integral form. In [32] (see also [13]), for general \(\text{Re}(a) > -1\), it was expressed in terms of a fifth Painlevé transcendent. Explicitly, it was found that
\begin{equation}
E_{\text{LUE}}^{n,a}(t) = \exp \left( \int_0^t \frac{ds}{s} U_V(s) \right),
\end{equation}
where \(U_V(s)\) satisfies the Jimbo-Miwa-Okamoto \(\sigma\)-form of the Painlevé V equation
\begin{equation}
(t\sigma'')^2 - (\sigma - t\sigma' + 2(\sigma')^2 + (\nu_0 + \nu_1 + \nu_2 + \nu_3)\sigma')^2 + 4(\nu_0 + \sigma')(\nu_1 + \sigma')(\nu_2 + \sigma')(\nu_3 + \sigma') = 0,
\end{equation}
with parameters
\begin{equation}
\nu_0 = \nu_1 = 0, \quad \nu_2 = n + a, \quad \nu_3 = n.
\end{equation}
Alternatively the conventional Painlevé V parameters are
\begin{equation}
\alpha = \frac{1}{2} a^2, \quad \beta = 0, \quad \gamma = -2n - a - 1, \quad \delta = -\frac{1}{2},
\end{equation}
and in terms of the Okamoto parameters they are
\begin{equation}
v_2 - v_1 = 0, \quad v_3 - v_1 = n + a, \quad v_4 - v_1 = n, \quad v_3 - v_4 = a.
\end{equation}
Because the eigenvalue density is strictly zero for \(\lambda < 0\), the neighbourhood of the smallest eigenvalue is referred to as the hard edge, and is denoted by HE\(_a\). As is consistent with (1.13), a well defined limit of (1.17) is obtained by the scaling \(t \mapsto t/4n\) and \(n \to \infty\). Thus [33]
\begin{equation}
\lim_{n \to \infty} E_{\text{LUE}}^{n,a}(t/4n) := E_{\text{HE}}^a(t) = \exp \left( \int_0^t \frac{ds}{s} U_{\text{III}}(s) \right),
\end{equation}
where \(U_{\text{III}}(t)\) satisfies the Jimbo-Miwa-Okamoto \(\sigma\)-form of the Painlevé III' equation
\begin{equation}
(t\sigma'')^2 - v_1 v_2 (\sigma')^2 + \sigma'(4\sigma' - 1)(\sigma - t\sigma') - \frac{1}{43}(v_1 - v_2)^2 = 0,
\end{equation}
with parameters
\begin{equation}
v_1 = v_2 = a,
\end{equation}
and subject to the boundary condition
\begin{equation}
U_{\text{III}}(t) \sim \frac{1}{2^{2a+2}\Gamma(a+1)\Gamma(a+2)} t^{a+1},
\end{equation}
Our interest in this paper is the distribution $P_{(2)}^{LUE_m,a}$ and its hard edge scaled limit. Analogous to (1.14), we see from (1.1) that
\begin{equation}
\lim_{n \to \infty} \frac{1}{16 n^2} P_{(2)}^{LUE_{n+2,a}}(x_1, x_2) = \frac{1}{n! c_{n+2,n+2+a}} e^{-x_1-x_2} (x_1-x_2)^2 (x_1 x_2)^a \times \int_{x_2}^{\infty} d\lambda_1 \ldots \int_{x_2}^{\infty} d\lambda_n \prod_{j=1}^{n} e^{-\lambda_j} \int_{x_1}^{\infty} \lambda_j^a (\lambda_j-x_1)^2 (\lambda_j-x_2)^2 \prod_{1 \leq j < k \leq n} (\lambda_k - \lambda_j)^2,
\end{equation}
where $x_1$ denotes the smallest eigenvalue and $x_2$ the second smallest eigenvalue. In [12], for $a$ integer, this was expressed as an $(a+2) \times (a+2)$ determinant. In the hard edge scaled limit this gave
\begin{equation}
\lim_{n \to \infty} \frac{1}{16 n^2} P_{(2)}^{LUE_{n+2,a}}(x_1, x_2) = \frac{1}{n! c_{n+2,n+2+a}} e^{-x_1-x_2} (x_1-x_2)^2 (x_1 x_2)^a \times \det \left[ \begin{array}{l} \int_{j=k}^{k+2} (\sqrt{x_2}) \left( \prod_{k=1}^{a+2} I_{j=1}^{j+2} (\sqrt{x_2}) \right) \prod_{k=1}^{a+2} I_{j=1}^{j+2} (\sqrt{x_2} - x_1) \end{array} \right].
\end{equation}
We seek a Painlevé type characterisation of (1.26) and its scaled limit, valid for general $\text{Re}(a) > -1$.

One use of knowledge of $P_{(2)}^{LUE_m,a}$ is the computation of the distribution of the spacing between the smallest and the second smallest eigenvalues. Denoting this $A_{n,a}$ by $P_{(2)}^{HE_m}(x_1, x_2)$, we have
\begin{equation}
A_{n,a}(y) := \int_{x_1}^{\infty} d\lambda_1 \ldots \int_{x_1}^{\infty} d\lambda_n \prod_{l=1}^{n} w(\lambda_l) \prod_{l=1}^{n} (\lambda_l-x_1) (\lambda_l-x_2) \prod_{1 \leq j < k \leq n} (\lambda_k - \lambda_j)^2,
\end{equation}
where $I$ denotes the support of the weight $w(\lambda)$. We have
\begin{equation}
P_{(2)}(x, x+y) = \frac{1}{c_{n+2,n+2+a}} e^{-(n+1)(x+y)-x y^2 [x(x+y)]^a D_n(-y, -y) [\lambda^2 (\lambda + x + y)^a e^{-\lambda} \chi > 0]},
\end{equation}
and
\begin{equation}
A_{n,a}(y) = \frac{y^2 e^y}{c_{n+2,n+2+a}} \int_{y}^{\infty} dt t^a (t-y)^a e^{-(n+2)t} D_n(-y, -y) [\lambda^2 (\lambda + t)^a e^{-\lambda} \chi > 0].
\end{equation}
These equations exhibit the occurrence of a deformation of the Laguerre weight

\[ w(x; t) := x^2(x + t)^ae^{-x}, \quad x \in \mathbb{R}_+. \]

This deformed weight actually interpolates between two Laguerre weights - when \( t \to 0 \) then we have the general parameter \( a + 2 \) case, whilst if \( t \to \infty \) in the sector \(-\pi < \arg(t) \leq \pi \) we have the special parameter situation with an exponent of 2. In fact virtually all of our analysis can be carried over to the more general situation where the exponent 2 is an arbitrary complex parameter suitably restricted.

We begin in Section 2 by revising appropriate results from orthogonal polynomial system theory and apply this to the particular deformed Laguerre weight (1.32). This allows us, in Section 3, to characterise the distributions (1.30) and (1.31) by a solution of the fifth Painlevé equation and its associated linear isomonodromic system (see Proposition 3.2). Section 4 is devoted to the determinant evaluations of those distributions for positive integer values of the parameter \( a \). We proceed in Section 5 to the study of the hard edge limits

\begin{equation}
\label{eq:1.33}
P_{(2)}^{\text{HE}_a}(x_1, x_2) := \lim_{n \to \infty} \frac{1}{16n^2} P_{(2)}^{\text{LUE}_{n,a}}(\frac{x_1}{4n}, \frac{x_2}{4n}),
\end{equation}

\begin{equation}
\label{eq:1.34}
A_a(z) := \lim_{n \to \infty} \frac{1}{4n} A_{n,a}(\frac{z}{4n}).
\end{equation}

It is found that these scaled distributions can be characterised by the solution of a certain Painlevé III equation and its associated linear isomonodromic system (see Propositions 5.2, 5.5 and Remark 5.3). In Section 6 this characterisation is used to obtain the high precision numerical values of statistical characteristics of \( A_a(z) \) for various integer values of \( a \) and for the values \( a = \pm 1/2 \), the latter being relevant to (1.6), (1.7) with the change of variable \( \lambda_j^2 \mapsto \lambda_j \). Let \( p_{(2)}^\pm(x_1, x_2) \) denote the scaled distribution of the eigenvalues \( e^{i\theta_1}, e^{i\theta_2}(\theta_1, \theta_2 > 0) \) closest to the origin in \( O^+(2n + 1) \) and \( O^+(2n) \) respectively. With the scaling chosen so that the bulk density is unity, it follows from (1.13) and (1.33) that

\begin{equation}
\label{eq:1.35}
p_{(2)}^\pm(x_1, x_2) = 4\pi^2 x_1 x_2 P_{(2)}^{\text{HE}_{\pm 1/2}}(\pi^2 x_1^2, \pi^2 x_2^2).
\end{equation}

Consequently

\begin{equation}
\label{eq:1.36}
A^\pm(y) := \int_0^\infty dx \, p_{(2)}^\pm(x, x + y)
= 4\pi^2 \int_0^\infty dx \, (x + y) P_{(2)}^{\text{HE}_{\pm 1/2}}(\pi^2 x^2, \pi^2 (x + y)^2).
\end{equation}

We use our results for \( P_{(2)}^{\text{HE}_{\pm 1/2}} \) to provide the high precision numerical values of statistical characteristics of \( A^\pm(y) \).
2. Orthogonal Polynomial System

2.1. Semi-classical Orthogonal Polynomials. Consider the general orthogonal polynomial system \( \{ p_n(x) \}^\infty_{n=0} \) defined by the orthogonality relations

\[
\int_I dx w(x) p_n(x) x^m = \begin{cases} 0 & 0 \leq m < n \\ h_n & m = n \end{cases},
\]

with \( I \) denoting the support of the weight \( w(x) \). We give special notation for the coefficients of \( x^n \) and \( x^{n-1} \) in \( p_n(x) \),

\[
p_n(x) = \gamma_n x^n + \gamma_{n,1} x^{n-1} + \ldots.
\]

The corresponding monic polynomials are then

\[
\pi_n(x) = \frac{1}{\gamma_n} p_n(x).
\]

It follows from (2.1) that

\[
\int_I dx w(x)(p_n(x))^2 = \gamma_n h_n,
\]

and thus for \( p_n(x) \) to be normalised as well as orthogonal we set \( \gamma_n h_n = 1 \). A consequence of the orthogonality relation is the three term recurrence relation

\[
a_{n+1} p_{n+1}(x) = (x - b_n) p_n(x) - a_n p_{n-1}(x), \quad n \geq 1,
\]

and we consider the set of orthogonal polynomials with initial values \( p_{-1} = 0 \) and \( p_0 = \gamma_0 \). The three term recurrence coefficients are related to the polynomial coefficients by [31], [14]

\[
a_n = \frac{\gamma_{n-1}}{\gamma_n}, \quad b_n = \frac{\gamma_{n,1}}{\gamma_n} - \frac{\gamma_{n+1,1}}{\gamma_{n+1}}, \quad n \geq 1,
\]

along with

\[
b_0 = -\frac{\gamma_{1,1}}{\gamma_1}, \quad a_0 = 0, \quad \gamma_{0,1} = 0.
\]

A well known consequence of (2.4) is the Christoffel-Darboux summation

\[
\sum_{j=0}^{n-1} p_j(x) p_j(y) = a_n \frac{[p_n(x)p_{n-1}(y) - p_{n-1}(x)p_n(y)]}{x - y}.
\]

Central objects in our probabilistic model are the Hankel determinants

\[
\Delta_n := \det[\mu_{j+k-2}]_{j,k=1,\ldots,n}, \quad n \geq 1, \quad \Delta_0 := 1,
\]

and

\[
\Sigma_n := \det \begin{bmatrix} \mu_0 & \cdots & \mu_{n-2} & \mu_n \\
\vdots & \ddots & \vdots & \vdots \\
\mu_{n-1} & \cdots & \mu_{2n-3} & \mu_{2n-1} \end{bmatrix}, \quad n \geq 1, \quad \Sigma_0 := 0,
\]
defined in terms of the moments \( \{ \mu_n \}_{n=0,1,\ldots,\infty} \) of the weight,

\[
(2.10) \quad \mu_n := \int_I dx \, w(x)x^n.
\]

We have integral representations for \( \Delta_n \)

\[
(2.11) \quad \Delta_n = \frac{1}{n!} \int_I dx_1 \ldots \int_I dx_n \prod_{l=1}^n w(x_l) \prod_{1 \leq j < k \leq n} (x_k - x_j)^2, \quad n \geq 1,
\]

and \( \Sigma_n \)

\[
(2.12) \quad \Sigma_n = \frac{1}{n!} \int_I dx_1 \ldots \int_I dx_n \prod_{l=1}^n w(x_l) \left( \sum_{j=1}^n x_j \right) \prod_{1 \leq j < k \leq n} (x_k - x_j)^2, \quad n \geq 1.
\]

The three-term recurrence coefficients are related to these determinants by standard result in orthogonal polynomial theory [31], [14]

\[
(2.13) \quad a_{n+1}^2 = \frac{\Delta_{n+1} \Delta_n - \Delta_n^2}{\Delta_n^2}, \quad n \geq 1,
\]

\[
(2.14) \quad b_n = \frac{\Sigma_{n+1}}{\Delta_{n+1}} - \frac{\Sigma_n}{\Delta_n}, \quad n \geq 0,
\]

\[
(2.15) \quad \gamma_n^2 = \frac{\Delta_n}{\Delta_{n+1}}, \quad n \geq 0,
\]

with initial values

\[
(2.16) \quad a_1^2 = \frac{\mu_0 \mu_2 - \mu_1^2}{\mu_0^2}, \quad b_0 = \frac{\mu_1}{\mu_0}, \quad \mu_0 \gamma_0^2 = 1.
\]

The orthogonal polynomials themselves also have a determinantal representation

\[
(2.17) \quad \sqrt{\Delta_n \Delta_{n+1}} p_n(x) = \det \begin{bmatrix} \mu_0 & \cdots & \mu_n \\ \vdots & \ddots & \vdots \\ \mu_{n-1} & \cdots & \mu_{2n-1} \\ 1 & \cdots & x^n \end{bmatrix}, \quad n \geq 1,
\]

and the integral representation

\[
(2.18) \quad \sqrt{\Delta_n \Delta_{n+1}} p_n(x) = \frac{1}{n!} \int_I dx_1 \ldots \int_I dx_n \prod_{l=1}^n w(x_l)(x-x_l) \prod_{1 \leq j < k \leq n} (x_k - x_j)^2.
\]

Another set of polynomial solutions to the three term recurrence relation are the associated polynomials \( \{ p_n^{(1)}(x) \}_{n=0}^{\infty} \), defined by

\[
(2.19) \quad p_n^{(1)}(x) := \int_I ds \, w(s) \frac{P_n(s) - p_n(x)}{s-x}, \quad n \geq 0.
\]

In particular these polynomials satisfy

\[
(2.20) \quad a_{n+1} p_n^{(1)}(x) = (x - b_n) p_{n-1}^{(1)}(x) - a_n p_{n-2}^{(1)}(x),
\]

with the initial conditions \( p_{-1}^{(1)}(x) = 0, \quad p_0^{(1)}(x) = \mu_0 \gamma_1 \). Note the shift by one decrement in comparison to the three-term recurrence (2.4) for the polynomials.
\{p_n(x)\}_{n=0}^{\infty}. We also need the definition of the moment generating function or Stieltjes function

\begin{align}
  f(x) & := \int_I ds \frac{w(s)}{x-s}, \quad x \notin I, \\
  & = \sum_{n=0}^{\infty} \frac{\mu_n}{x^{n+1}}, \quad x \notin I, \quad x \to \infty.
\end{align}

We define non-polynomial associated functions \{\epsilon_n(x)\}_{n=0}^{\infty} by

\begin{align}
  \epsilon_n(x) & := f(x)p_n(x) - p^{(1)}_{n-1}(x),
\end{align}

which also satisfy the three term recurrence relation (2.4), namely

\begin{align}
  a_{n+1}\epsilon_{n+1}(x) & = (x - b_n)\epsilon_n(x) - a_n\epsilon_{n-1}(x),
\end{align}

subject to the initial values \epsilon_{-1}(x) = 0, \epsilon_0(x) = \gamma_0f(x). The associated functions have an integral representation analogous to (2.18)

\begin{align}
  \sqrt{\Delta_n\Delta_{n+1}}\epsilon_n(x)
  & = \frac{1}{(n+1)!} \int_I dx_1 \cdots \int_I dx_{n+1} \prod_{i=1}^{n+1} \frac{w(x_i)}{x - x_i} \prod_{1 \leq j < k \leq n+1} (x_k - x_j)^2, \quad x \notin I.
\end{align}

The polynomials and their associated functions satisfy the Casoratian relation

\begin{align}
  p_n(x)\epsilon_{n-1}(x) - p_{n-1}(x)\epsilon_n(x) & = \frac{1}{a_n}, \quad n \geq 1.
\end{align}

Extending (2.2) and (2.5) we have

\begin{align}
  p_n(x) & = \gamma_n \left[ x^n - \left( \sum_{i=0}^{n-1} b_i \right) x^{n-1} \\
  & \quad + \left( \sum_{0 \leq i < j < n} b_i b_j - \sum_{i=1}^{n-1} a_i^2 \right) x^{n-2} + O(x^{n-3}) \right],
\end{align}

valid for \( n \geq 1 \), while for the associated functions

\begin{align}
  \epsilon_n(x) & = \gamma_n^{-1} \left[ x^{-n-1} + \left( \sum_{i=0}^{n} b_i \right) x^{-n-2} \\
  & \quad + \left( \sum_{0 \leq i < j \leq n} b_i b_j + \sum_{i=1}^{n+1} a_i^2 \right) x^{-n-3} + O(x^{-n-4}) \right],
\end{align}

valid for \( n \geq 0 \).

**Proposition 2.1** ([6],[4],[24]). Let

\begin{align}
  \frac{1}{w(x)} \frac{d}{dx} w(x) & = \frac{2V(x)}{W(x)},
\end{align}

for \( V, W \) irreducible. The orthogonal polynomials and associated functions satisfy a system of coupled first order linear differential equations with respect to \( x \) \((' \equiv d/dx)\)

\[
W p'_n = (\Omega_n - V)p_n - a_n \Theta_n p_{n-1}, \quad n \geq 1, \tag{2.30}
\]

\[
W p'_{n-1} = a_n \Theta_{n-1} p_n - (\Omega_n + V)p_{n-1}, \quad n \geq 0, \tag{2.31}
\]

for certain coefficient functions \( V(x), W(x), \Theta_n(x), \Omega_n(x) \). The associated functions \( \epsilon_n, \epsilon_{n-1} \) satisfy precisely the same set of equations.

If we define the \( 2 \times 2 \) matrix variable

\[
Y_n(x; t) = \begin{pmatrix}
p_n(x) & \epsilon_n(x) \\
p_{n-1}(x) & \epsilon_{n-1}(x)
\end{pmatrix}
\]

then the above coupled system can be written as

\[
\frac{d}{dx} Y_n(x) = \frac{1}{W(x)} \begin{pmatrix}
\Omega_n(x) - V(x) & -a_n \Theta_n(x) \\
a_n \Theta_{n-1}(x) & -\Omega_n(x) - V(x)
\end{pmatrix} Y_n(x) \tag{2.33}
\]

It follows that the coefficient functions are specified by

\[
\Theta_n = W [\epsilon_n p'_n - \epsilon'_n p_n] + 2V \epsilon_n p_n, \quad n \geq 0, \quad \Theta_{-1} = 0, \tag{2.34}
\]

\[
\Omega_n = a_n W [\epsilon_{n-1} p'_n - \epsilon'_n p_{n-1}] + a_n V [\epsilon_n p_{n-1} + \epsilon_{n-1} p_n], \quad n \geq 1, \quad \Omega_0 = 0. \tag{2.35}
\]

**Proposition 2.2** ([24]). The coefficient functions arising in Proposition 2.1 satisfy the recurrence relations

\[
(\Omega_{n+1} - \Omega_n)(x - b_n) = W + a_{n+1}^2 \Theta_{n+1} - a_n^2 \Theta_{n-1}, \quad n \geq 0, \tag{2.36}
\]

\[
\Omega_{n+1} + \Omega_n = (x - b_n) \Theta_n, \quad n \geq 0 \tag{2.37}
\]

We will find it necessary to study the zeros of the orthogonal polynomial \( p_n(x) \) which we denote \( \{x_{1,n} < \ldots < x_{j,n} < \ldots < x_{n,n}\} \). They have an electrostatic interpretation as the equilibrium positions of the mobile unit charges, and there is a set of equations governing these equilibrium positions known as the Bethe Ansatz equations.

**Proposition 2.3** ([16]). The zeros \( \{x_{j,n}\}_{j=1}^n \) of the polynomial \( p_n(x) \) satisfy the coupled functional equations

\[
2 \sum_{k \neq j} \frac{1}{x_{j,n} - x_{k,n}} = \frac{\Theta'_n(x_{j,n})}{\Theta_n(x_{j,n})} - \frac{W'(x_{j,n}) + 2V(x_{j,n})}{W(x_{j,n})}, \tag{2.38}
\]

for all \( 1 \leq j \leq n \).
One can also represent many useful quantities in terms of sums over the zeros and we illustrate this with an example. Firstly the consecutive ratios of the orthogonal polynomials have a partial fraction decomposition

\begin{equation}
\frac{p_{n-1}(x)}{p_n(x)} = \frac{1}{a_n} \sum_{j=1}^{n} \frac{W(x_{j,n})}{\Theta_n(x_{j,n})} \frac{1}{x - x_{j,n}},
\end{equation}

along with

\begin{equation}
a_n^2 = -\sum_{j=1}^{n} \frac{W(x_{j,n})}{\Theta_n(x_{j,n})}.
\end{equation}

Of particular relevance to our application are the semi-classical class of orthogonal polynomial systems [25] defined by the property that \( V(x) \) and \( W(x) \) in (2.29) are polynomials in \( x \). The zeros of \( W(x) \) define finite singularities of the system of ordinary differential equations (2.30), (2.31) and will feature prominently in this study. Let \( x_r \) be such a point with \( r \geq 1 \). Then at \( x_r \) the relations (2.36) and (2.37) can be combined and integrated to yield

\begin{equation}
\Omega_n^2(x_r) - V^2(x_r) = a_n^2 \Theta_n(x_r) \Theta_{n-1}(x_r), \quad n \geq 1.
\end{equation}

In fact, at a given finite singular point \( x_r \), we can deduce the following bi-linear identities, that factorise the one above.

**Corollary 2.1.** The coefficient functions evaluated at a finite singular point \( x_r \) are related to evaluations of the orthogonal polynomials and associated functions by the relations

\begin{align}
\Omega_n(x_r) + V(x_r) &= 2a_n V(x_r) p_n(x_r) \epsilon_{n-1}(x_r), \quad n \geq 1, \\
\Omega_n(x_r) - V(x_r) &= 2a_n V(x_r) p_{n-1}(x_r) \epsilon_n(x_r), \quad n \geq 1, \\
\Theta_n(x_r) &= 2V(x_r) p_n(x_r) \epsilon_n(x_r). \quad n \geq 0
\end{align}

From the theory of Uvarov the following general result for (1.29) is known.

**Proposition 2.4 ([35]).** The quantity \( D_n(x,x)[w(\lambda)] \), defined by the equal argument form of (1.29), is evaluated in terms of the polynomials \( p_n(x) \) orthogonal with respect to \( w(x) \) and coefficients \( \gamma_n, \Delta_n \) of this system as

\begin{equation}
D_n(x,x)[w(\lambda)] = \frac{\Delta_n}{\gamma_n \gamma_{n+1}} [p_n(x)p'_{n+1}(x) - p_{n+1}(x)p'_n(x)].
\end{equation}

**Proof.** This is a specialisation of Uvarov’s general result to the case \( k = 0 \) and \( l = 2 \) where the integral \( D_n(x_1,x_2)[w(\lambda)] \) is a Hankel determinant with respect to the weight \( w_{0,2}(x) \), defined by

\begin{equation}
w_{0,2}(x) dx = dp_{0,2}(x), \quad \rho_{0,2}(x) = \int^{x} (s - x_1)(s - x_2)w(s)ds.
\end{equation}
The non-confluent form of the corresponding identity states that
\[
D_n(x_1, x_2)[w(\lambda)] = (\Delta_{n+2}\Delta_n)^{1/2} \frac{\det[p_{n+k-1}(x_j)]_{j,k=1,2}}{x_2 - x_1},
\]
and the result follows under the confluence $x_2 \to x_1$. \qed

We see from (2.45) that our main task is to obtain appropriate characterisations of the orthogonal polynomials and their derivatives associated with the weight (1.32).

2.2. Deformed Laguerre Orthogonal Polynomials. As we noted in the introduction we see the appearance of a deformed Laguerre weight (1.32) which is a member of the semi-classical class with the polynomials $V, W$ in (2.29) specified by
\[
2V(x; t) = -x^2 + (a + 2 - t)x + 2t, \quad W(x; t) = x(x + t),
\]
and has finite singularities at $x = 0, -t$. The moments have the simple evaluation
\[
\mu_n(t) = t^{a+n+3} \Gamma(n + 3) U(n + 3, a + n + 4; t), \quad n \geq 0, \quad |\text{arg}(t)| < \pi,
\]
where $U(\alpha, \gamma; z)$ is the confluent hypergeometric that is not analytic at $z = 0$. The moments can be written as a sum of two parts one of which is analytic and the other non-analytic about $t = 0$
\[
\mu_n(t) = \Gamma(a + n + 3) \left\{ \begin{array}{l}
\frac{1}{2} F_1(-a; -a - n - 2; t) \\
(1)(n+3) \frac{1}{2} F_1(a+1; a+n+4; t) \end{array} \right.
\]
where we have to exclude the cases $a \in \mathbb{Z}_{\geq 0}$.

Within the semi-classical class the coefficient functions $\Theta_n(x), \Omega_n(x)$ are polynomials with degree fixed independently of the index $n$. In particular we can relate these polynomials to the coefficients of the orthogonal polynomials themselves.

**Proposition 2.5.** The coefficient functions are
\[
\Theta_n(x) = 2n + a + 3 - t - b_n - x, \quad n \geq 0,
\]
\[
\Omega_n(x) = \frac{1}{2} x^2 + \frac{1}{2} (2n + a + 2 - t)x + (n + 1)t - a_n^2 - \frac{\gamma_n}{\bar{\gamma}_n}, \quad n \geq 1.
\]

**Proof.** From the theory of [24] we note that the degrees of the coefficient functions are $\deg \Theta_n \leq \max\{\deg W - 2, \deg V - 1\} = 1$ and $\deg \Omega_n \leq \max\{\deg W - 1, \deg V - 1, \deg \Theta_n - 1, \deg U + 1\} = 2$. To obtain explicit forms for these we use the definitions (2.34) and (2.35) and the large $x \to \infty$ expansions of the polynomials and associated functions given in (2.27) and (2.28). The first equalities in (2.51) and (2.52) then follow. \qed
We will also find it convenient to make the following definitions motivated by the above result,

\[ \theta_n := 2n + a + 3 - t - b_n, \]
\[ \kappa_n := (n + 1)t - a_n^2 - \frac{\gamma_{n+1}}{\gamma_n}. \]

**Proposition 2.6.** The spectral derivatives of the polynomials \( p_n(x) \) are \( (\equiv \partial/\partial x) \)

\[ x(x + t)p'_n = (nx + \kappa_n - t)p_n - a_n(\theta_n - x)p_{n-1} \]
\[ x(x + t)p'_{n-1} = a_n(\theta_{n-1} - x)p_n - (-x^2 + (n + a + 2 - t)x + t + \kappa_n)p_{n-1} \]

*Proof.* This follows from the general form of the spectral derivatives (2.30) and (2.30), along with the explicit particular forms (2.51) and (2.52). \( \square \)

**Proposition 2.7.** The deformation derivatives of the orthogonal polynomials are \( (\equiv \partial/\partial t) \)

\[ t(x + t)p_n = (n + 1)t - \kappa_n - \frac{1}{2}(x + t)(\theta_n + t) \]
\[ t(x + t)p'_{n-1} = -a_n(\theta_{n-1} + t)p_n + \left[ \frac{1}{2}(x + t)(\theta_{n-1} + t) + \kappa_n - (n + a + 1)t \right] p_{n-1} \]

*Proof.* We will opt to establish this relation directly from the orthonormality conditions on the polynomials

\[ \int_I dx \ W(x)p_n(x)p_{n-i}(x) = \delta_{i,0}, \quad 0 \leq i \leq n. \]

Differentiating this with respect to \( t \) leaves us with the relation

\[ 0 = a \int_I dx \ W(x)p_n(x)p_{n-i}(x) + \int_I dx \ W(x)p_n p_{n-i} + \int_I dx \ W(x)p_n \dot{p}_{n-i}, \]

where use of the logarithmic derivative of \( W(x) \) has been made. Now we employ

\[ \dot{p}_{n-i} = \frac{\gamma_{n-i}}{\gamma_n} p_{n-i} + \Pi_{n-i-1}, \]

to write the last term of (2.59) as

\[ \int_I dx \ W(x)p_n \dot{p}_{n-i} = \frac{\gamma_{n-i}}{\gamma_n} \delta_{i,0} = \frac{\gamma_n}{\gamma_n} \int_I dx \ W(x)p_n p_{n-i}. \]
Considering the first term of (2.59) we note
\[
\int_I dx w(x)p_n(x)p_{n-1}(x)\frac{p_{n-1}(x) - p_{n-1}(-t)}{x + t} = \int_I dx w(x)p_n(x)\frac{p_{n-1}(x) - p_{n-1}(-t)}{x + t} + p_{n-1}(-t) \int_I dx w(x)\frac{p_n(x)}{x + t},
\]
\[
= p_{n-1}(-t) \int_I dx w(x)\frac{p_n(x)}{x + t},
\]
\[
= -p_{n-1}(-t)\epsilon_n(-t).
\]
But we can recast \(p_{n-1}(-t)\) as
\[
p_{n-1}(-t) = \sum_{j=0}^{n} p_{n-j}(-t)\delta_{i,j},
\]
\[
= \sum_{j=0}^{n} p_{n-j}(-t) \int_I dx w(x)p_{n-j}(x)p_{n-1}(x),
\]
\[
= \int_I dx w(x)p_{n-1}(x) \sum_{j=0}^{n} p_{n-j}(-t)p_{n-j}(x),
\]
\[
= \int_I dx w(x)p_{n-1}(x) \sum_{j=0}^{n} p_j(-t)p_j(x).
\]
Combining these we deduce that
\[
\int_I dx w(x)p_{n-1}(x) \left\{ \dot{p}_n(x) + \gamma_n p_n(x) - a\epsilon_n(-t) \sum_{j=0}^{n} p_j(-t)p_j(x) \right\} = 0,
\]
for \(i = 0, \ldots, n\). The factor in curly brackets in the integrand must be a polynomial in \(x\) with degree less than or equal to \(n\), and yet is orthogonal to all polynomials \(p_j\) for \(j = 0, \ldots, n\), and thus must be identically zero. This gives us our first form for the deformation derivative of \(p_n\),
\[
(2.60) \quad \dot{p}_n(x) = -\gamma_n p_n(x) + a\epsilon_n(-t) \sum_{j=0}^{n} p_j(-t)p_j(x).
\]
Equating coefficients of \(p_n(x)\) in this relation we find
\[
(2.61) \quad 2\gamma_n = a\epsilon_n(-t).
\]
Furthermore using the Christoffel-Darboux formula (2.7), and the above equation, we can express this derivative solely as a linear combination of \(p_n\) and \(p_{n-1}\)
\[
(2.62) \quad \dot{p}_n(x) = a\epsilon_n(-t) \left[ \frac{1}{2} p_n(-t) + a_n \frac{p_{n-1}(-t)}{x + t} \right] p_n(x)
\]
\[
- a a_n \epsilon_n(-t) \frac{p_n(-t)}{x + t} p_{n-1}(x).
\]
Using the bilinear product relations (2.43) and (2.44) we arrive at the result (2.57). The second of the two relations can be found by shifting \( n \rightarrow n - 1 \) in (2.62) and using the three term recurrence relation. \( \square \)

In the matrix formulation the spectral derivative take the particular form

\[
\partial_x Y_n(x; t) = \left\{ A_\infty + \frac{A_0}{x} + \frac{A_t}{x + t} \right\} Y_n(x; t).
\]  

The residue matrices are explicitly given by

\[
A_0 = \frac{1}{t} \begin{pmatrix}
\kappa_n - t & -a_n \theta_n \\
-a_n \theta_{n-1} & -\kappa_n - t
\end{pmatrix}, \quad \chi_0 = 0, -2
\]

\[
A_t = \frac{1}{t} \begin{pmatrix}
(n+1)t - \kappa_n & a_n(\theta_n + t) \\
-a_n(\theta_{n-1} + t) & \kappa_n - (n+a+1)t
\end{pmatrix}, \quad \chi_t = 0, -a
\]

\[
A_\infty = \begin{pmatrix}
0 & 0 \\
0 & 1
\end{pmatrix}
\]

Our linear system has two regular singularities at \( x = 0, -t \) and an irregular singularity at \( x = \infty \) with Poincaré index 1. In this formulation the deformation derivative is

\[
\partial_t Y_n(x; t) = \left\{ B + \frac{A_t}{x + t} \right\} Y_n(x; t)
\]

\[
B = \frac{1}{2t} \begin{pmatrix}
-\theta_n - t & 0 \\
0 & \theta_{n-1} + t
\end{pmatrix}
\]

As we will see in Section 3.2 (2.63) and (2.67) form the monodromy preserving system corresponding to the fifth Painlevé equation with a form equivalent to that discussed by Jimbo [18], in contrast to other forms studied in [19], [7], [8] or [17].

**Corollary 2.2.** The polynomial coefficients satisfy the following coupled, first order mixed deformation derivative and difference equations

\[
\frac{2t}{a_n} \frac{\dot{a}_n}{a_n} = 2 + b_{n-1} - b_n, \quad n \geq 1,
\]

\[
tb_n = a_n^2 - a_{n+1}^2 + b_n, \quad n \geq 0,
\]

with the initial \( t = 0 \) values for \( b_n \) and \( a_n^2 \) given by (2.87) and (2.88) respectively. This system of differential equations is equivalent to the Schlesinger equations.

**Proof.** There are several methods of proof available here. The first is using the general result of [24], expressing the deformation derivatives of the polynomial coefficients in terms of a sum of the coefficient functions over the movable finite
singular points

\[
\frac{\dot{a}_n}{a_n} = \frac{1}{2} \sum_{r=1}^{m} \frac{\Theta_n(x_r) - \Theta_{n-1}(x_r)}{W'(x_r)} \dot{x}_r, \quad n \geq 1,
\]

\[
\dot{b}_n = \sum_{r=1}^{m} \frac{\Omega_{n+1}(x_r) - \Omega_n(x_r)}{W'(x_r)} \dot{x}_r, \quad n \geq 1.
\]

The only finite singular point contributing here is \( x = -t \).

Alternatively one can find these derivatives from the polynomial derivatives by examining selected coefficients. For example by considering the coefficients of \( x^{n-1} \) in (2.60) we have

\[
\dot{\gamma}_{n,1} = \frac{1}{2} a \epsilon_n(-t)p_n(-t)\gamma_{n,1} + a \epsilon_n(-t)p_{n-1}(-t)\gamma_{n-1},
\]

and therefore

\[
\left( \frac{\dot{\gamma}_{n,1}}{\gamma_{n}} \right) = a \epsilon_n(-t)p_{n-1}(-t).
\]

Consequently, using (2.5), we find that

\[
\frac{\dot{a}_n}{a_n} = \frac{1}{2} a \left[ \epsilon_{n-1}(-t)p_{n-1}(-t) - \epsilon_n(-t)p_n(-t) \right],
\]

\[
\dot{b}_n = a \left[ a_n \epsilon_n(-t)p_{n-1}(-t) - a_{n+1} \epsilon_{n+1}(-t)p_n(-t) \right],
\]

which are identical to (2.69) and (2.70) respectively.

**Corollary 2.3.** The recurrences for the polynomial coefficients are

\[
\alpha_{n+2}^2 - \alpha_n^2 = 2t + b_{n+1} [2n + a + 6 - t - b_{n+1}] - b_n [2n + a + 2 - t - b_n], \quad n \geq 1.
\]

and

\[
\alpha_{n+1}^2 [2n + a + 5 - t - b_n - b_{n+1}] - \alpha_n^2 [2n + a + 1 - t - b_{n-1} - b_n]
\]

\[
= -b_n (b_n + t), \quad n \geq 1.
\]

The initial data \( b_0(t) \) and \( \alpha_1^2(t) \) are given by (2.16) with the evaluation of the moments (2.49).

**Proof.** The first relation (2.77) follows by substituting the explicit forms for the coefficient functions, (2.51) and (2.52), into the relation (2.37) and requiring equality of the polynomials in \( x \). Equality is trivial for \( x^2 \) and \( x^1 \), whilst the non-trivial equality for \( x^0 \) gives (2.77). The second relation (2.78) follows from the same procedure applied to the recurrence relation (2.36) and again the only nontrivial result occurs for the \( x^0 \) part. \( \square \)
This result can also be recovered from a specialisation of work in [5]. In their system of monic orthogonal polynomials the three term recurrence coefficients $\beta_n, \gamma_n$ (not to be confused with our use of these symbols subsequently) are related to ours by

\[(2.79)\quad \beta_n = b_n, \quad \gamma_n = a_n^2.\]

Equation (39) of this work implies

\[(2.80)\quad \gamma_{n+1} + \gamma_n = (2n + 3)t + (2n + a + 4 - t)\beta_n + 2 \sum_{k=0}^{n-1} \beta_k - \beta_n^2,\]

and differencing this once leads directly to (2.77). In addition their equation (40) implies

\[(2.81)\quad \gamma_{n+1}\beta_{n+1} = (2n + a + 5 - t - \beta_n)\gamma_{n+1} + 2 \sum_{k=1}^n \gamma_k + \sum_{k=0}^n \beta_k (\beta_k + t).\]

Again differencing this once one finds precisely (2.78).

A check of the above results can be made when $t \to 0$ whilst all parameters are kept fixed since this, as noted before, corresponds to the Laguerre weight with parameter $a + 2$. Therefore we have

\[(2.82)\quad a_n^2(0) = n(n + a + 2),\]
\[(2.83)\quad b_n(0) = 2n + a + 3,\]
\[(2.84)\quad \Delta_n(0) = \prod_{j=1}^n j!\Gamma(j + a + 2).\]

In our normalisation we have

\[(2.85)\quad p_n(x; 0) = (-1)^n \left\{ \frac{n!}{\Gamma(n + a + 3)} \right\}^{1/2} L_n^{(a+2)}(x),\]

where $L_n^{(\alpha)}(x)$ are the standard associated Laguerre polynomials of degree $n$ and index $\alpha$. We see that the spectral derivative equations (2.55) and (2.56) reduce to the standard expressions for the derivative of the Laguerre polynomials, the coefficients in the right-hand sides of the deformation derivatives (2.57) and (2.58) vanish, the recurrence relations (2.77) and (2.78) are identically satisfied, and the right-hand sides of (2.69) and (2.70) are zero.

In fact we will need to develop expansions about $t = 0$ in order to characterise our quantities as particular solutions of difference and differential equations in Section 3. To this end we have the following result.
Proposition 2.8. For fixed \( n \) and \( a \not\in \mathbb{Z}_{\geq 0} \) the Hankel determinant (2.8) with the weight (1.32) has the expansion about \( t = 0 \)

\[
\Delta_n(t) = \Delta_n(0) \left\{ 1 + \frac{a}{a+2} nt \right. \\
+ \frac{1}{4} \left. \frac{a}{a+2} \left( \frac{(a-1)(n+1)}{a+1} + \frac{(a+1)(n-1)}{a+3} \right) nt^2 + O(t^3) \right. \\
- \left. \frac{2\Gamma(a+1)}{\Gamma(a+3)\Gamma^2(a+4)} \frac{\Gamma(a+n+3)}{\Gamma(n)} t^{a+3} (1 + O(t)) + O(t^{2a+6}) \right\},
\]

with \( |\arg(t)| < \pi \). Consequently, under the same conditions, the three-term recurrence coefficients have the expansions about \( t = 0 \)

\[
b_n(t) = 2n + a + 3 - \frac{a}{a+2} t + \frac{2a(2n + a + 3)}{(a+3)(a+2)^2(a+1)} t^2 + O(t^3) \\
+ \frac{2}{(a+2)(a+1)\Gamma^2(a+3)} \frac{\Gamma(a+n+3)}{\Gamma(n+1)} t^{a+3} (1 + O(t)) + O(t^{2a+6}),
\]

and

\[
a_n^2(t) = n(n+a+2) \left\{ 1 - \frac{2a}{(a+3)(a+2)^2(a+1)} t^2 + O(t^3) \right. \\
- \left. \frac{2}{(a+1)\Gamma(a+3)\Gamma(a+4)} \frac{\Gamma(a+n+2)}{\Gamma(n+1)} t^{a+3} (1 + O(t)) + O(t^{2a+6}) \right\}.
\]

Proof. We adopt the method of expanding the Hankel determinant by expanding the moments to leading order

\[
\mu_n(t) = \Gamma(a+n+3) + a\Gamma(a+n+2)t + \frac{1}{2} a(a-1)\Gamma(a+n+1)t^2 + O(t^3) \\
+ (-1)^{n+1} \frac{\Gamma(a+1)\Gamma(n+3)}{\Gamma(a+n+4)} t^{n+a+3} + O(t^{n+a+4}),
\]

as \( t \to 0 \) using (2.50). The determinant can be expanded to leading orders in \( t, t^{a+3} \) and the resulting determinants evaluated using the identity [26]

\[
det(\Gamma(z_k + j))_{j,k=0,\ldots,n-1} = \prod_{j=0}^{n-1} \Gamma(z_j) \prod_{0 \leq j < k \leq n-1} (z_k - z_j),
\]

where \( \{z_0, \ldots, z_{n-1}\} \) is an arbitrary sequence not necessarily in arithmetic progression. \( \square \)

We conclude this section by noting some identities relating the polynomial coefficients and the zeros of the polynomials. Firstly we give the Bethe Ansatz equations for the zeros of the deformed Laguerre orthogonal polynomials which can be directly deduced from Proposition 2.3.
Corollary 2.4. The zeros $x_{j,n}$ of the deformed Laguerre orthogonal polynomials $p_n(x)$ satisfy the functional equations

$$
\frac{3}{x_{j,n}} + \frac{a+1}{x_{j,n}+t} - \frac{1}{x_{j,n} - \theta_n} + 2 \sum_{k \neq j} \frac{1}{x_{j,n} - x_{k,n}} = 1, \quad 1 \leq j \leq n.
$$

According to the electrostatic interpretation, the terms of (2.91) can be interpreted in the following way - the first is the interaction of the mobile unit charge at $x_{j,n}$ with the fixed charge of size 3 at the singularity $x = 0$, the second with the fixed charge of size $a + 1$ at the singularity $x = -t$, the third with a fixed charge of size $-1$ at the apparent singularity $x = \theta_n$, the fourth the mutual repulsion with the other mobile charges and the term on the right-hand side is the linear confining potential. From the partial fraction decomposition (2.39) specialised to the arguments $x = 0, -t$ we have the summation identities.

Proposition 2.9. The following summations over the zeros have the explicit evaluations

$$
\frac{\kappa_n - (n+1)t}{\theta_n + t} = \sum_{j=1}^{n} \frac{x_{j,n}}{\theta_n - x_{j,n}},
$$

$$
\frac{\kappa_n - t}{\theta_n} = \sum_{j=1}^{n} \frac{t + x_{j,n}}{\theta_n - x_{j,n}},
$$

$$
\frac{1}{\theta_n + t} \left( n + \frac{\kappa_n - t}{\theta_n} \right) = \sum_{j=1}^{n} \frac{1}{\theta_n - x_{j,n}},
$$

$$
\alpha_n^2 = \sum_{j=1}^{n} \frac{x_{j,n}(x_{j,n} + t)}{x_{j,n} - \theta_n}.
$$

In addition we can characterise the motion of the zeros with respect to the deformation variable.

Proposition 2.10. The zeros $x_{j,n}(t)$ satisfy the differential equation with respect to $t$

$$
t \dot{x}_{j,n} = \frac{\theta_n + t}{\theta_n - x_{j,n}} x_{j,n}.
$$

Proof. This follows by equating

$$
\dot{p}_n(x) = \frac{\gamma_n}{\gamma_n} \sum_{j=1}^{n} \frac{\dot{x}_{j,n}}{x - x_{j,n}},
$$

and (2.57), and then employing (2.39) along with (2.48), (2.51).

3. Difference and Differential Equations

3.1. Difference Equations. In the first subsection we derive an alternative difference system in terms of the new variables $\theta_n(t), \kappa_n(t)$ as specified by (2.53), (2.54).
Proposition 3.1. The auxiliary functions $\theta_n(t), \kappa_n(t)$ satisfy a system of coupled first order recurrence relations

$$\kappa_{n+1} + \kappa_n = \theta_n(\theta_n + t - 2n - a - 3), \quad n \geq 0,$$

$$\frac{\theta_n}{\theta_n + t} \frac{\theta_{n-1}}{\theta_{n-1} + t} = \frac{(\kappa_n - t)(\kappa_n + t)}{[\kappa_n - (n + a + 1)t][\kappa_n - (n + 1)t]}, \quad n \geq 1. \tag{3.1}$$

The initial values $\theta_0$ and $\kappa_0$ are given by

$$\theta_0(t) = -2t \int_0^\infty dx e^{-x} x(t + x)^a, \quad \kappa_0 = t. \tag{3.3}$$

Proof. The first of the recurrence relations (2.77) can be exactly summed and the result is

$$a_{n+1}^2 + a_n^2 = (2n + 3)t + (2n + a + 4 - t)b_n + 2 \sum_{i=0}^{n-1} b_i - b_n^2. \tag{3.4}$$

Recalling the second relation of (2.5) the summation appearing here can done by recasting the equation in terms of the new variables and yields

$$\kappa_{n+1} + \kappa_n = -\theta_n b_n, \tag{3.5}$$

which is (3.1). The second member of the coupled set is most easily found from the general relation (2.41) evaluated at the finite singular points $x = 0, -t$ and employing the new variables. These two key identities are

$$(\kappa_n + t)(\kappa_n - t) = a_n^2 \theta_n \theta_{n-1}, \tag{3.6}$$

$$[\kappa_n - (n + a + 1)t][\kappa_n - (n + 1)t] = a_n^2(\theta_n + t)(\theta_{n-1} + t). \tag{3.7}$$

The ratio of these two identities yields the relation (3.2). \qed

There are other recurrence relations which will be used subsequently, and the first is

$$a_n^2(\theta_n + \theta_{n-1} + t) = -(2n + a + 2)\kappa_n + [n^2 + (n + 1)(a + 2)]t. \tag{3.8}$$

This follows from the subtraction of (3.6) from (3.7). The second relation

$$a_{n+1}^2 - a_n^2 - b_n - t = 2\kappa_n + b_n \theta_n, \tag{3.9}$$

is derived by writing the definition of $\kappa_{n+1} - \kappa_n$ in terms of the old variables and then employing (3.1). The last relation

$$a_{n+1}^2 \theta_{n+1} - a_n^2 \theta_{n-1} = b_n(2\kappa_n + b_n \theta_n), \tag{3.10}$$

is a consequence of (2.78) along with the use of (3.9).
As a consequence of relations (3.6) from (3.7) we have
\[
\frac{\theta_n}{\theta_n + t} \left[ n + \frac{\kappa_n - t}{\theta_n} \right] = \theta_n \frac{\kappa_n - t}{\theta_n} - \frac{\kappa_n - (n + 1)t}{\theta_n + t},
\]
\[
= \frac{\theta_n}{t} \left\{ a_n^2 \frac{\theta_{n-1}}{\kappa_n + t} - a_n^2 \frac{\theta_{n-1} + t}{\kappa_n - (n + a + 1)t} \right\},
\]
\[
= -\frac{\kappa_n - t}{\kappa_n - (n + a + 1)t} \left[ n + a + 2 + \frac{\kappa_n + t}{\theta_{n-1}} \right],
\]
(3.11)
and
\[
n + \frac{\kappa_n - t}{\theta_n} + \left\{ n + a + 2 + \frac{\kappa_n + t}{\theta_{n-1}} \right\} \bigg|_{n \to n+1} = \theta_n + t.
\]
(3.12)

3.2. Reduction to Painlevé V. Here we will identify the fifth Painlevé system as the solution to our system of equations characterising the deformed Laguerre orthogonal polynomial system. This is most simply seen in terms of the new variables \(\theta_n, \kappa_n\) rather than the basic orthogonal polynomial variables \(a_n, b_n\).

**Proposition 3.2.** The auxiliary quantities \(\theta_n(t), \kappa_n(t)\) satisfy the coupled first order ordinary differential equations
\[
t \dot{\theta}_n = 2\kappa_n + \theta_n(2n + a + 3 - t - \theta_n),
\]
(3.13)
and
\[
t \dot{\kappa}_n = \left( \frac{1}{\theta_n + t} + \frac{1}{\theta_n} \right) \kappa_n^2 + \left( 2n + a + 3 - (2n + a + 2) \frac{t}{\theta_n + t} \right) \kappa_n
\]
\[- [n^2 + (n + 1)(a + 2)t - \frac{t^2}{\theta_n} + (n + 1)(n + a + 1) \frac{t^2}{\theta_n + t}].
\]
(3.14)
Equations (3.13) and (3.14) can be solved in terms of the fifth Painlevé system
\[
\theta_n = t \frac{q}{1 - q}, \quad \kappa_n = t(1 + qp),
\]
(3.15)
where \(q, p\) are the Hamiltonian variables of the Okamoto P\(V\) [27] system with the parameters
\[
\alpha = \frac{a^2}{2}, \quad \beta = -2, \quad \gamma = -(2n + a + 3), \quad \delta = -\frac{1}{2},
\]
or
\[
v_2 - v_1 = -2, \quad v_3 - v_1 = n + a, \quad v_4 - v_1 = n, \quad v_3 - v_4 = a.
\]
The solutions satisfy the boundary value data at \(t = 0\)
\[
\theta_n(t) \bigg|_{t \to 0} = -\frac{2}{a + 2} t - \frac{2a(2n + a + 3)}{(a + 3)(a + 2)^2(a + 1)} t^2 + O(t^3)
\]
\[- \frac{2}{(a + 2)(a + 1)\Gamma^2(a + 3)} \frac{\Gamma(a + n + 3)}{\Gamma(n + 1)} t^{a+3} (1 + O(t)) + O(t^{2a+6}),
\]
(3.17)
and

\[
\kappa_n(t) = \frac{2n + a + 2}{a + 2}t + \frac{4n(a + 2)a}{(a + 3)(a + 2)(a + 1)}t^2 + O(t^3)
\]
\[
+ \frac{2}{(a + 2)(a + 1)\Gamma(a + 3)} \frac{\Gamma(a + n + 3)}{\Gamma(n)} t^{a+3} (1 + O(t)) + O(t^{2a+6}),
\]

provided \(a \not\in \mathbb{Z}_{\geq 0}\) and \(|\arg(t)| < \pi\).

**Proof.** If we employ (3.9) in (2.70) and then substitute for \(b_n\) in terms of \(\theta_n\) then the result is (3.13). Let us define the shorthand notation

\[
\Gamma_n := \frac{\gamma_{n+1}}{\gamma_n}.
\]

Furthermore when the deformation derivative (2.70) is summed on the free index the result is

\[
t \dot{\Gamma}_n = a_n^2 + \Gamma_n.
\]

Now if compute the deformation derivative of \(\kappa_n\) and use (2.69) along with (3.21) we arrive at

\[
t \dot{\kappa}_n = \kappa_n - a_n^2 (\theta_n - \theta_{n-1}).
\]

Now the idea is to eliminate \(a_n^2\) and \(\theta_{n-1}\), which appear in (3.22), through use of the recurrence relations. Equation (3.2) is a linear equation for \(\theta_{n-1}\) in terms of the unshifted variables and the solution can be substituted into (3.8) yielding a linear relation for \(a_n^2\) in terms of unshifted variables. Then both solutions can be substituted into (3.22) and the result is (3.14). One can easily verify that the transformation to the Hamiltonian variables \(q, p\) (3.15) with the parameters (3.16) yields the Hamilton equations of motion for the Hamiltonian in [27]. \(\square\)

**Remark 3.1.** A few remarks can now be made regarding the identification of the recurrences (3.1) and (3.2). This is different in appearance from the discrete integrable equations that arose in the study of the Laguerre unitary ensemble [13] which were explicitly identified with the system in the Sakai scheme, with the rational surface \(D_5^{(1)} \rightarrow E_6^{(1)}\), and has a continuous limit of Painlevé IV. In fact we find that the variables of the latter system can be expressed in terms of our own

\[
x_n = \frac{\kappa_n + \theta_n(n + a + 1 - t - \theta_n)}{\theta_n + t}, \quad y_n = -\frac{t}{\theta_n},
\]

and it is clear that one cannot transform (3.1) and (3.2) into this system using such a transformation. Also the two systems arise as different Schlesinger-type transformations - in our case as a sequence where \(\alpha_0 \mapsto \alpha_0 + 1, \alpha_2 \mapsto \alpha_2 - 1\) whereas in the other case as \(\alpha_0 \mapsto \alpha_0 + 1, \alpha_3 \mapsto \alpha_3 - 1\).
Remark 3.2. In [13] two fundamental quantities were studied - the \( \tau \)-function \( \tau[n](t) \) and its logarithmic derivative the \( \sigma \)-function \( V_n(t; a, \mu) \). Their relation to the objects of the present work are

\[
\tau[n] = c(n,a)n!e^{-nt-n^2-n(a+4)}\Delta_n(t),
\]

where \( c(n,a) \) is an unspecified constant and

\[
V_n(t; a, 2) = -nt - 4n + t\frac{d}{dt}\log\Delta_n(t).
\]

We also note that

\[
V_n(t; a, 2) = \Gamma_n + n(n + a - 2 - t)
\]

and consequently

\[
\Gamma_n = -n(n + a + 2) + t\frac{d}{dt}\log\Delta_n(t)
\]

\[
b_n = 2n + a + 3 + t\frac{d}{dt}\log\frac{\Delta_n(t)}{\Delta_{n+1}(t)}
\]

Remark 3.3. The new variable \( \Gamma_n(t) \) possess an expansion as \( t \to 0 \) which can be directly found from (2.86) and (2.87)

\[
n(n + a + 2) + \Gamma_n(t) = \frac{a}{a + 2}nt - \frac{2n(n + a + 2)a}{(a + 3)(a + 2)^2(a + 1)}t^2 + O(t^3)
\]  

\[- \frac{2}{(a + 2)(a + 1)\Gamma(a + 3)\Gamma(a + 4)}\frac{\Gamma(a + n + 3)}{\Gamma(n)}t^{a+3}(1 + O(t)) + O(t^{2a+6}),
\]

again provided \( a \not\in \mathbb{Z}_{\geq 0} \).

Remark 3.4. The use of Proposition 3.2 is in the computation of the orthogonal polynomials in (2.4) corresponding to the weight (1.32). For this we note from (2.53) that

\[
b_n = 2n + a + 3 - t - \theta_n,
\]

while (2.54) together with (2.5), (2.6) show

\[
a_n^2 = (n + 1)t - \frac{\gamma_{n,1}}{\gamma_n} - \kappa_n
\]

\[
= (n + 1)t + \sum_{j=0}^{n-1} b_j - \kappa_n,
\]

(the quantity \( a_n \) is positive, so the positive square root of this equation is to be taken). Further, it follows from (2.13) and (2.5) that

\[
\frac{1}{\gamma_0^2\gamma_1^2\ldots\gamma_{n-1}^2} = \Delta_n,
\]

where each \( \gamma_j \) is the coefficient of \( x^j \) in \( p_j(x) \) as specified by (2.2). All terms in the equation (2.45) for \( D_n(x,x) \) are then known, and the task is then to compute the integral as required by (1.31).
An alternative system of coupled first order ordinary differential equations which will be used for scaling to the hard edge is given in the following proposition.

**Proposition 3.3.** The variables $\theta_n(t), \Gamma_n(t)$ satisfy the coupled first order ordinary differential equations

\begin{align*}
(3.34) \quad t \dot{\theta}_n &= \theta_n - \left\{ \theta_n^4 - 2(2n + a + 2 - t)\theta_n^3 \\
+ [4\Gamma_n + (2n + a + 2 - t)^2 - 4(n + 1)t]\theta_n^2 \\
+ 4t[\Gamma_n + n(n + a + 2 - t) + a + 2 - t]\theta_n + 4t^2 \right\}^{1/2},
\end{align*}

and

\begin{align*}
(3.35) \quad t \dot{\Gamma}_n &= (n + 1)t + \frac{1}{2} \theta_n(2n + a + 2 - t - \theta_n) + \left\{ \theta_n^4 - 2(2n + a + 2 - t)\theta_n^3 \\
+ [4\Gamma_n + (2n + a + 2 - t)^2 - 4(n + 1)t]\theta_n^2 \\
+ 4t[\Gamma_n + n(n + a + 2 - t) + a + 2 - t]\theta_n + 4t^2 \right\}^{1/2}.
\end{align*}

**Proof.** We proceed in a series of steps. Firstly we use (3.6) to solve for $\theta_{n-1}$ in terms of $\theta_n, \kappa_n$ and $\Gamma_n$. In the second step we substitute this solution for $\theta_{n-1}$ into (3.7) and solve the following quadratic equation for $\kappa_n$ in terms of $\theta_n$ and $\Gamma_n$,

\begin{equation}
(3.36) \quad \kappa_n^2 + \theta_n(2n + a + 2 - t - \theta_n)\kappa_n + [(n + 1)t - \Gamma_n]\theta_n(\theta_n + t) - [n^2 + (n + 1)(a + 2)]t\theta_n - t^2 = 0.
\end{equation}

The choice of the sign of the square-root branch follows from the expansions (3.18) and (3.29) on one hand, and on the other hand noting that as $t \to 0$

\begin{equation}
(3.37) \quad \left\{ \theta_n^4 - 2(2n + a + 2 - t)\theta_n^3 \\
+ [4\Gamma_n + (2n + a + 2 - t)^2 - 4(n + 1)t]\theta_n^2 \\
+ 4t[\Gamma_n + n(n + a + 2 - t) + a + 2 - t]\theta_n + 4t^2 \right\}^{1/2}
= \frac{2a(2n + a + 3)}{(a + 3)(a + 2)(a + 1)} t^2 + O(t^3).
\end{equation}

In the final step we use these solutions for $\theta_{n-1}$ and $\kappa_n$ in (3.22) and (3.21). □

In preparation for the hard edge scaling limit we need to make evaluations of the polynomials at the finite singular points. Firstly considering $x = 0$ we note that

\begin{equation}
(3.38) \quad \frac{\pi_n(0)}{\pi_{n-1}(0)} = a_n \frac{p_n(0)}{p_{n-1}(0)} = \frac{\kappa_n + t}{\theta_{n-1}} = \frac{a_n^2}{\theta_n} \frac{\theta_n}{\kappa_n - t},
\end{equation}

as follows immediately from (2.55). Furthermore we also have

\begin{equation}
(3.39) \quad t(\log \pi_n(0)) = t(\log p_n(0)) + \frac{1}{2}(\theta_n + t) = n + \frac{\kappa_n - t}{\theta_n},
\end{equation}
which follows from (2.57) and the above result. The corresponding result for the polynomial ratio at $x = -t$

\[
\frac{p_n(-t)}{p_{n-1}(-t)} = \frac{1}{a_n} \frac{\kappa_n - (n + a + 1)t}{\theta_{n-1} + t} = a_n \frac{\theta_n + t}{\kappa_n - (n + 1)t}.
\]

After [1] we define the orthogonal polynomial ratios

\[
Q_n(x; t) := \frac{p_n(x; t)}{p_n(0; t)},
\]

because we are interested in the scaling properties of the orthogonal polynomial system at the edge of their interval of orthogonality, $x = 0$. Then (2.4) and Propositions 2.6 and 2.7 can be translated into the following three corollaries.

**Corollary 3.1.** The three-term recurrence for \{Q_n\} system is

\[
b_n(Q_n + Q_{n-1} - 2Q_n) + (b_n + 2\frac{\kappa_n - t}{\theta_n})(Q_{n+1} - Q_{n-1}) + 2xQ_n = 0.
\]

**Corollary 3.2.** The spectral derivatives of $Q_n, Q_{n-1}$ are

\[
x(x + t)Q_n' = nxQ_n + (\kappa_n - t)\left[Q_n - Q_{n-1} + \frac{x}{\theta_n}Q_{n-1}\right],
\]

\[
x(x + t)Q_{n-1}' = x[x - (n + a + 2 - t)]Q_{n-1} + (\kappa_n + t)\left[Q_n - Q_{n-1} - \frac{x}{\theta_{n-1}}Q_n\right].
\]

**Corollary 3.3.** The deformation derivatives of $Q_n, Q_{n-1}$ are

\[
t(x + t)\dot{Q}_n = -x(n + \frac{\kappa_n - t}{\theta_n})Q_n + (\kappa_n - t)\frac{\theta_n + t}{\theta_n}[Q_{n-1} - Q_n],
\]

\[
t(x + t)\dot{Q}_{n-1} = x[n + a + 2 + \frac{\kappa_n + t}{\theta_{n-1}}]Q_{n-1} + (\kappa_n + t)\frac{\theta_{n-1} + t}{\theta_{n-1}}[Q_{n-1} - Q_n].
\]

As we noted earlier the polynomial ratio $Q_n(x; t)$ has a product representation

\[
Q_n(x; t) = \prod_{j=1}^{n} \left(1 - \frac{x}{x_{j,n}}\right),
\]

where again $x_{j,n}$ is the $j$-th zero of the polynomial. We can use this fact to compute sums of the inverse powers of the zeros from the above differential equations.

**Proposition 3.4.** The increment of the sum of the reciprocals of the zeros going from $n - 1$ to $n$ is given by

\[
3t \left\{-\sum_{j=1}^{n} \frac{1}{x_{j,n}} + \sum_{j=1}^{n-1} \frac{1}{x_{j,n-1}}\right\} = n + \frac{\kappa_n - t}{\theta_n} + n + a + 2 + \frac{\kappa_n + t}{\theta_{n-1}} - t.
\]
Proof. The required increment of the sum of the reciprocals of the zeros is the order \( x \) term in the expansion of \( \psi_n(x; t) := Q_n(x; t)/Q_{n-1}(x; t) \) about \( x = 0 \), and this can evaluated from the same expansion of the differential equation for \( \psi_n \) with respect to \( x \). This latter differential equation is easily found from (3.43) and (3.44) and is

\[
x(x+t)\psi_n' = \frac{x - \theta_n}{\theta_n}(\kappa_n - t) + (2\kappa_n + x[-x+2n+a+2-t])\psi_n + \frac{x - \theta_{n-1}}{\theta_{n-1}}(\kappa_n + t)\psi_n^2.
\]

At the other finite singular point, \( x = -t \), we have as a consequence of these corollaries

\[
\frac{Q_n(-t; t)}{Q_{n-1}(-t; t)} = \frac{\kappa_n - t}{\kappa_n - (n+1)t} \frac{\theta_n + t}{\theta_n} = \frac{\kappa_n - (n+a+1)t}{\kappa_n + t} \frac{\theta_n - 1}{\theta_n + t},
\]

and

\[
\frac{d}{dt}Q_n(-t; t) = \frac{t - \kappa_n - n\theta_n}{\theta_n(\theta_n + t)}Q_n(-t; t).
\]

Note that the derivative with respect to \( t \) in the latter equation is a total derivative.

To complete our preparations for the hard edge scaling we need to identify two polynomial variables that will scale to independent variables in the scaling limit. The first is the orthogonal polynomial ratio \( Q_n \), and for the second a number of choices could be made but a simple choice is

\[
R_n := Q_n - Q_{n-1}.
\]

**Corollary 3.4.** The spectral derivatives of \( Q_n, R_n \) are

\[
x(x+t)Q_n' = x \left( n + \frac{\kappa_n - t}{\theta_n} \right) Q_n + (\kappa_n - t) \frac{\theta_n - x}{\theta_n} R_n,
\]

\[
x(x+t)R_n' = x \left[ n + \frac{\kappa_n - t}{\theta_n} + n + a + 2 + \frac{\kappa_n + t}{\theta_{n-1}} - x - t \right] Q_n
\]

\[
+ \left[ -x \left( n + \frac{\kappa_n - t}{\theta_n} \right) + x(x+t) - (a+2)x + 2t \right] R_n.
\]

**Proof.** This follows from Corollary 3.2. \( \square \)

**Corollary 3.5.** The deformation derivatives of \( Q_n, R_n \) are

\[
t(x+t)\dot{Q}_n = -x \left( n + \frac{\kappa_n - t}{\theta_n} \right) Q_n - (\kappa_n - t) \frac{\theta_n + t}{\theta_n} R_n,
\]

\[
t(x+t)\dot{R}_n = -x \left[ n + \frac{\kappa_n - t}{\theta_n} + n + a + 2 \right] Q_n
\]

\[
+ \left[ x \left( n + a + 2 + \frac{\kappa_n + t}{\theta_{n-1}} \right) + (\kappa_n + t) \frac{\theta_{n-1} + t}{\theta_{n-1}} - (\kappa_n - t) \frac{\theta_n + t}{\theta_n} \right] R_n.
\]
Proof. This follows from Corollary 3.3. □

3.3. Inequalities and Bounds. A key step in proving our hard edge scaling limits will be bounds on the variables $\theta_n, \kappa_n$ and some auxiliary quantities. The first step is the following result.

**Lemma 3.1.** The variables $\theta_n(t), \kappa_n(t)$ satisfy the inequalities

\[
\frac{\theta_n + t}{\kappa_n - (n+1)t} < \frac{\theta_n}{\kappa_n - t} < 0, \tag{3.57}
\]

\[
\frac{\theta_n - 1}{\kappa_n + t} < \frac{\theta_n - 1 + t}{\kappa_n - (n+a+1)t} < 0, \tag{3.58}
\]

for all positive, real and bounded $t$ and $n \geq 1$.

Proof. That the ratios given in (3.57) and (3.58) are negative is a consequence of the fact that the polynomial $p_n(x)$ evaluated on the negative real axis, i.e. exterior to the interval of orthogonality, has a fixed sign. Specifically $(-1)^n p_n(-y) > 0$ for real, positive $y$. Using the ratio relations (3.38) and (3.40) we have the upper bounds. From the Christoffel-Darboux formula (2.7) at equal arguments we note that

\[
p_{n-1}(x)p'_n(x) - p'_n(x)p_{n-1}(x) > 0, \tag{3.59}
\]

and from the above $p_{n-1}(x)p_n(x) < 0$ for $x \in -\mathbb{R}_+$ we conclude

\[
\frac{p_n(x)}{p_{n-1}(x)} < \frac{p'_n(x)}{p'_{n-1}(x)}, \tag{3.60}
\]

under the conditions on $x$. Integrating this inequality from 0 to $-y \in -\mathbb{R}_+$ we arrive at

\[
\frac{p_n(-y)}{p_{n-1}(-y)} < \frac{p_n(0)}{p_{n-1}(0)} < 0. \tag{3.61}
\]

Then identifying these ratios with (3.38) and (3.40) in the case $y = t$ leads to the relative inequalities. □

The above set of inequalities must all apply simultaneously and we see in fact that it implies restrictions on the variables $\theta_n, \kappa_n$.

**Lemma 3.2.** For bounded $t \in \mathbb{R}_+$ and $n \geq 0$ the variables $\theta_n, \kappa_n$ satisfy inequalities which place them in one of three cases, as illustrated in Figure 1.

Case I:

\[
0 < \theta_n, \tag{3.62}
\]

\[
\theta_n(\theta_n + t - 2n - a - 3) + t < \kappa_n < t - n\theta_n, \tag{3.63}
\]

Case II:

\[
-t \leq \theta_n \leq 0, \tag{3.64}
\]

\[
t - n\theta_n \leq \kappa_n \leq t + \max\{nt, \theta_n(\theta_n + t - 2n - a - 3)\} \leq (n+1)t, \tag{3.65}
\]
Proof. From the residue formula (2.44) we recall that $\Theta_n(0) = \theta_n = 2t\rho_n(0)\epsilon_n(0)$ and $\Theta_n(-t) = \theta_n + t = -at\rho_n(-t)\epsilon_n(-t)$. As we noted in the proof of Lemma 3.3.

**Figure 1.** A pictorial form of the inequalities taking the example of $a = 1/2$, $n = 2$ and $t = 5/3$, which illustrates the generic situation for $n + a + 3 > t$.

**Case III:**

(3.66) \[ \theta_n < -t, \]

(3.67) \[ (n+1)t < \kappa_n < t - n\theta_n. \]

Proof. From the three inequalities implied by (3.57) we see that $\theta_n \geq 0$ according as $\kappa_n \leq t$, $\theta_n + t \geq 0$ according as $\kappa_n \leq (n+1)t$, and $\theta_n + t \geq 0$ according as $\kappa_n \leq t - n\theta_n$. In (3.58) we make the replacement $n \mapsto n+1$ and employ (3.1) to eliminate $\kappa_{n+1}$. This inequality now reads

(3.68) \[ -t - \frac{\kappa_n - (n+1)t}{\theta_n + t} < -\frac{\kappa_n - t}{\theta_n} < b_n = 2n + a + 3 - t - \theta_n. \]

Consequently these three inequalities imply $\theta_n \geq 0$ according as $\kappa_n - t \geq \theta_n(\theta_n + t - 2n - a - 3)$, $\theta_n + t \geq 0$ according as $\kappa_n - (n+1)t \geq (\theta_n + t)(\theta_n - 2n - a - 3)$, and $\theta_n(\theta_n + t) \geq 0$ according as $\kappa_n \leq \theta_n(\theta_n + t - n) + t$. Combining these sets of inequalities leads to the three cases above. \[\Box\]

We see that Case II applies in our situation.

**Lemma 3.3.** For all $n$ and $t \in \mathbb{R}_+$ we have $-t \leq \theta_n \leq 0$ and $t - n\theta_n \leq \kappa_n \leq (n+1)t$.

Proof. From the residue formula (2.44) we recall that $\Theta_n(0) = \theta_n = 2t\rho_n(0)\epsilon_n(0)$ and $\Theta_n(-t) = \theta_n + t = -at\rho_n(-t)\epsilon_n(-t)$. As we noted in the proof of Lemma
(3.1) it is immediate from the integral representations of the polynomials and their associated functions, (2.18) and (2.25), that \((-1)^n p_n(-t) \geq 0\) and \((-1)^{n+1} \epsilon_n(-t) \geq 0\) for all real \(t \geq 0\). This places \(\theta_n\) in the range applying to Case II. □

We can also draw some conclusions concerning the zeros of the orthogonal polynomials which will be important subsequently.

**Corollary 3.6.** Each zero \(x_{j,n}(t)\) is a monotonically decreasing function of \(t\) and interpolates between the Laguerre zero with \(t = 0\) and exponent \(a + 2\) and the Laguerre zero with \(t = \infty\) and exponent 2,

\[
x_{j,n}(0) > x_{j,n}(t) > x_{j,n}(\infty) > 0,
\]

for all \(1 \leq j \leq n\) and bounded \(t > 0\).

**Corollary 3.7.** The following bounds on the reciprocal sums over the zeros hold

\[
\sum_{j=1}^{n} \frac{1}{x_{j,n}(t)} \leq \frac{n}{2},
\]

\[
\sum_{j=1}^{n} \frac{1}{x_{j,n}(t) + t} \leq \frac{n}{a + 3}.
\]

**Proof.** From the two-sided bound on \(\theta_n\) we can deduce

\[
\frac{1}{x_{j,n}(t) + t} \leq \frac{1}{x_{j,n}(t) - \theta_n} \leq \frac{1}{x_{j,n}(t)}.
\]

Employing this in the Bethe Ansatz (2.91) summed over \(j\) we arrive at the above bounds. □

4. **Special Case** \(a \in \mathbb{Z}_{\geq 0}\)

Our evaluation of the distribution function in terms of the fifth Painlevé system is with all three free parameters variable in some sense - one is fixed in this application at a positive integer, one is the index \(n \in \mathbb{Z}\) and the remaining one is \(a \in \mathbb{C}\). Up to this point we have studied in some depth the recurrence relations with respect to \(n\) while \(a\) has been left arbitrary other than being restricted because of the existence considerations. From the point of view of the Painlevé theory it is quite natural that the transcendental objects become classical when \(a \in \mathbb{Z}\) for either positive or negative subsets of the integers. In particular it is expected that the \(\tau\) functions in the theory will have Hankel determinantal forms of classical function entries with a rank dependent on \(a\). It is these cases which have been studied in the past [12],[11] using methods which transform the integral into the determinantal representations and then employ confluent Vandermonde identities.
Proposition 4.1. When \( a \in \mathbb{Z}_{>0} \) we have the evaluation for the Hankel determinant

\[
\Delta_n(t) = \frac{c_{n+1,n+1+a}}{(n+a)!} \det [L_{n+a+1-j-k}^{j+k-1-a}(-t)]_{j,k=1,...,a},
\]

and for \( a = 0 \)

\[
\Delta_n(t) = \frac{c_{n+1,n+1}}{n!}.
\]

Proof. In [12] Eq. (3.18) (after correcting) states

\[
\Delta_n(t) = \frac{c_{n+1,n+1+a}}{(n+a)!} (-1)^{(a-1)/2} \det [D_x^{j+k-2} L_{n+a-1}^{-(a-3)}(x) | x = -t]_{j,k=1,...,a},
\]

where \( D_x := d/dx \). Using the Laguerre polynomial identity

\[
D^m x L_n^{(a)}(x) = (-1)^m L_{n-m}^{(a+m)}(x), \quad m \in \mathbb{Z}_{>0},
\]

with the proviso \( L_n^{(a)}(x) = 0 \) for \( n < 0 \), we arrive at (4.1). \( \square \)

As a consequence of the relations (3.28) and (3.27) the variables \( \theta_n(t), \Gamma_n(t) \) will have \( a \times a \) determinant forms, and in particular for \( a = 0 \)

\[
\theta_n(t) = -t, \quad \kappa_n(t) = (n+1)t, \quad \Gamma_n(t) = -n(n+2).
\]

The orthogonal polynomials also have determinantal forms of the following type.

Proposition 4.2. When \( a \in \mathbb{Z}_{>0} \) the orthogonal polynomials are given by

\[
\sqrt{\Delta_n} p_n(x; t) = (-1)^{n+a+\lfloor \frac{a+1}{2} \rfloor} a! \cdots (n+a)! (n+1)! \cdot (x+t)^{-a} \det \left[ L_{n+a+1-j-k}^{j+k-1-a}(x) \right]_{j,k=1,...,a+1},
\]

and for \( a = 0 \)

\[
p_n(x; t) = (-1)^n \left( \frac{1}{(n+2)(n+1)} \right)^{1/2} L_n^{(2)}(x).
\]

If \( a > n+1 \) we note that \( L_{n+1-a}^{(a+1)}(-t) = 0 \). Consequently, under the same condition, the polynomial ratio is given by

\[
Q_n(x; t) = \left( \frac{t}{x+t} \right)^a \frac{\det \left[ L_{n+a+1-k}^{j+k-1-a}(x) \right]_{j,k=1,...,a+1}}{\det \left[ L_{n+a+2-j-k}^{j+k-1-a}(-t) \right]_{j,k=1,...,a+1}}.
\]
Proof. Starting with the integral representation (2.18) we follow the procedures used in [12]. Taking one factor of the squared product of differences we write it like

\[(4.9) \prod_{1 \leq j < k \leq n} (x_k - x_j) = \frac{1}{\prod_{l=0}^{n-1} c_l} \det[L_{k-1}^{(\alpha)}(x_j)]_{j,k=1,\ldots,n},\]

using the Vandermonde identity and where \(c_n\) is the leading coefficient of \(L_n^{(\alpha)}(x)\) and \(\alpha\) is a parameter to be fixed later. Of the remaining factors in the integrand we write

\[(4.10) \prod_{j=1}^{n} (x_j + t)(x - x_j) \prod_{1 \leq j < k \leq n} (x_k - x_j) = \frac{(-1)^{a(n+1)}}{\prod_{l=0}^{a-1} b_l \prod_{l=0}^{n-1} c_l} (x + t)^{-a} \times \det \begin{bmatrix} L_{k-1}^{(\alpha)}(x_j) & j=1,\ldots,n \\ L_{k-1}^{(\alpha)}(x) & k=1,\ldots,N \\ D_{y}^{-1}L_{k-1}^{(\alpha)}(y) & y=-t, j=1,\ldots,a, k=1,\ldots,N \end{bmatrix},\]

where the confluent Vandermonde identity has been used and \(N = n + a + 1\).

Reassembling the integral with these two factors, then expanding the determinant in (4.9) we multiply each of \(n\) factors into the determinant of (4.10). Making use of the antisymmetry of the row ordering in the first \(n\) rows of the determinant we can perform the \(n\) integrals as long as we choose \(\alpha = 2\). Then

\[(4.11) \sqrt{\Delta_n \Delta_{n+1}} p_n(x; t) = \frac{(-1)^{a(n+1)}}{\prod_{l=0}^{a-1} b_l \prod_{l=0}^{n-1} c_l} \prod_{j=0}^{n-1} \frac{(j+2)!}{j!} (x + t)^{-a} \times \det \begin{bmatrix} L_{k+1}^{(\alpha)}(x) & k=1,\ldots,a+1 \\ D_{y}^{-1}L_{n+k-1}^{(\alpha)}(y) & y=-t, j=1,\ldots,a, k=1,\ldots,a+1 \end{bmatrix}.\]

Using the identities

\[(4.12) L_n^{(\alpha-1)}(x) = L_n^{(\alpha)}(x) - L_{n-1}^{(\alpha)}(x),\]

along with elementary column operations, and then identity (4.4) we are lead to (4.6). The evaluation for the polynomial ratio (4.8) is a simple consequence of the first evaluation along with

\[(4.13) L_{n+a+1-k}^{(-a+1+k)}(0) = \binom{n+2}{n+a+1-k}.\]
Proposition 4.3. When $a \in \mathbb{Z}_{\geq 0}$ the deformed Hankel determinant is given by

\begin{equation}
D_n(x, x) = c_{n+2, n+2+a} (-1)^{\frac{1}{2}(a+1)(a+2)} (x + t)^{-2a} \times \det \left[ \begin{array}{cc}
L_{n+a+3-j-k}^{(j+k-1-a)} (-t) \\
L_{n+a+3-j-k}^{(j+k-1-a)} (x)
\end{array} \right]_{j=1, \ldots, a}^{j=1, \ldots, a+2}.
\end{equation}

Proof. The quantity $D_n(x, x)$ was essentially computed in [12] in Eq. (3.20), which can be recast as

\begin{equation}
D_n(x, x) = c_{n+2, n+2+a} (-1)^{\frac{1}{2}(a+1)(a+2)} (x + t)^{-2a} \times \det \left[ \begin{array}{cc}
D_u^{j+k-2} L_{n+a+1}^{-(a-1)} (u) |_{u=-t} \\
D_u^{j+k-2} L_{n+a+1}^{-(a-1)} (u) |_{u=x}
\end{array} \right]_{j=1, \ldots, a}^{j=1, \ldots, a+2}.\]
\end{equation}

Then (4.14) follows by application of the identity (4.4). □

As a consequence the distribution of the first eigenvalue spacing is

\begin{equation}
A_{n,a}(y) = (-1)^{\frac{1}{2}(a+1)(a+2)} y^2 e^y \int_y^{\infty} dt \, t^a (t - y)^{-a} e^{-(n+2)t} \times \det \left[ \begin{array}{cc}
L_{n+a+3-j-k}^{(j+k-a-1)} (-t) \\
L_{n+a+3-j-k}^{(j+k-a-1)} (y)
\end{array} \right]_{j=1, \ldots, a}^{j=1, \ldots, a+2},
\end{equation}

for $a = 1, 2, 3, \ldots$ and

\begin{equation}
A_{n,0}(y) = \frac{1}{n+2} y^2 e^{-(n+1)y} \left[ L_{n+1}^{(1)} (-y) L_{n-1}^{(3)} (-y) - (L_n^{(2)} (-y))^2 \right],
\end{equation}

for $a = 0$.

5. Hard Edge Scaling

5.1. General Case. We define new scaled spectral variables $s, z$ by

\begin{equation}
t = \frac{s}{4n}, \quad x = -\frac{z}{4n},
\end{equation}

in the triangular domain $s > z > 0$ and study the scaling of the finite distribution (1.31) as the polynomial degree $n \to \infty$. What is required here is not just the asymptotic scaling of the orthogonal polynomial coefficients but also of the polynomials themselves in the neighbourhood of an endpoint of the interval of orthogonality. For the deformed Laguerre polynomials this would be a generalisation of the asymptotics of Hilb’s type for the Laguerre polynomials as found in Szegő [31]

\begin{equation}
e^{x/2,x^{1/2}}_n (-x) = \frac{\Gamma(n + 1)}{n!(2^n M)^{\frac{1}{2}}} I_\mu (\sqrt{M} x) + O(n^{\frac{1}{2} - \frac{1}{2}}),
\end{equation}

where $\mu = -\frac{3}{2}$.
with $M = 4n + 2\mu + 2$ as $n \to \infty$ and $J_\mu(z)$ the standard modified Bessel function. Despite a resurgence in activity around these questions, especially the use of Riemann-Hilbert techniques on these problems, there are no results available for our particular problem. A general review of the asymptotics of orthogonal polynomials can be found in [23], and an introduction to the Riemann-Hilbert approach to the asymptotics is the chapter in [22]. However to establish the nature of and the existence of limits for our variables we do not require such techniques.

**Lemma 5.1.** Under the scaling of the (5.1) $4n\theta_n(t)$ and $\kappa_n(t)$ are bounded for all real positive $t$ and $n \geq 1$.

**Proof.** From the result of Lemma 3.3 we see that

$$-s \leq 4n\theta_n(s/4n) \leq 0, \quad s/4n - 4n\theta_n(s/4n) \leq \kappa_n(s/4n) \leq (n + 1)s/4n,$$

and the assertion follows. \qed

**Corollary 5.1.** Under the above conditions

$$n + \kappa_n(t) - t \left|_{t=s/4n} \right. = O(1), \quad \text{as } n \to \infty.$$

**Proof.** The above lemma states that

$$\theta_n(s/4n) = O(1), \quad \text{as } n \to \infty,$$

and we find as a consequence that also

$$\varepsilon := n(n + a + 2) - s/4 + \Gamma_n(s/4n) = O(1), \quad \text{as } n \to \infty.$$

The discriminant $D$ appearing in the workings of Proposition 3.3 can then be written as

$$D^2 := \theta_n^4 - 2(2n + a + 2 - t)\theta_n^3 + [4(\varepsilon - t) + (a + 2 - t)^2 - 4nt]\theta_n^2 + 4t[\varepsilon + a + 2 - t]\theta_n + 4t^2,$$

and therefore

$$\left. \frac{D(t)}{\theta_n(t)} \right|_{t=s/4n} = O(1), \quad \text{as } n \to \infty.$$

From the formula for $\kappa_n$ in Proposition 3.3 we note that

$$n + \frac{\kappa_n(t) - t}{\theta_n} = -\frac{a + 2}{2} - t \frac{1}{\theta_n} + \frac{1}{2}(\theta_n + t) - \frac{D}{2\theta_n},$$

and the result then follows. \qed

**Proposition 5.1.** For bounded $s \in \mathbb{R}_+$ under the scaling (5.1) the variables $\theta_n(t)$ and $\Gamma_n(t)$ converge to limits in the following manner

$$\lim_{n \to \infty} 4n\theta_n(t)\left|_{t=s/4n} \right. = \mu(s),$$

$$\lim_{n \to \infty} n(n + a + 2) + \Gamma_n(t)\left|_{t=s/4n} \right. = \nu(s).$$
The variable $\kappa_n(t)$ converges like

$$\lim_{n \to \infty} \kappa_n(t)|_{t=s/4n} = -\frac{1}{4} t(s).$$

Proof. Firstly we note that

$$\eta := a_n^2(t) - n(n + a + 2)\big|_{t=s/4n} = O(1), \quad \text{as } n \to \infty,$$

as follows from (5.6). Starting with (3.8) we see that

$$\theta_n - \theta_{n-1} = 2\theta_n + t + \frac{(2n + a + 2)\kappa_n - [n^2 + (n + 1)(a + 2)]t}{a_n^2},$$

so we require bounds on the difference of the last two terms on the right-hand side.

$$n\theta_n - (n-1)\theta_{n-1} = \frac{n}{a_n^2} \left\{ (2n + a + 2 + \eta)t + 2(n + a + 2)\theta_n \left( n + \frac{\kappa_n - t}{\theta_n} \right) \right\},$$

and thus

$$n\theta_n - (n-1)\theta_{n-1} = \frac{n}{a_n^2} \left\{ (2n + a + 2 + \eta)t + 2(n + a + 2)\theta_n \left( n + \frac{\kappa_n - t}{\theta_n} \right) \right\}.$$

Thus we have shown

$$n\theta_n(t) - (n-1)\theta_{n-1}(t)|_{t=s/4n} = O(n^{-1}), \quad \text{as } n \to \infty.$$

Now we can write the quantity of interest

$$n\theta_n(t)|_{t=s/4n} - (n-1)\theta_{n-1}(t)|_{t=s/4n(n-1)} = [n\theta_n(t) - (n-1)\theta_{n-1}(t)]|_{t=s/4n}$$

$$+ (n-1)\theta_{n-1}(t)|_{t=s/4n - (n-1)\theta_{n-1}(t)|_{t=s/4(n-1)}},$$

so we require bounds on the difference of the last two terms on the right-hand side.

Let $t = s/4n$ and $t_> = s/4(n-1)$. Because $\theta_n(t)$ is continuously differentiable and its derivative is given by (3.13)

$$(n-1) |\theta_{n-1}(t) - \theta_{n-1}(t_>)|$$

$$\leq (n-1)(t_> - t) \max_{u \in (t, t_>)} |\theta_{n-1}(u)|,$$

$$\leq \frac{s}{4n} \max_{u \in (t_>, t)} |\theta_{n-1}(u)|,$$

$$\leq \frac{s}{4n} \max_{u \in (t_>, t)} u^{-1} |2\kappa_n(u) + \theta_{n-1}(u)(2n + a + 1 - u - \theta_{n-1}(u))|,$$

$$\leq \frac{s}{4n} \max_{u \in (t_>, t)} u^{-1} |\theta_{n-1}(u)|$$

$$\times \left| 2 \left( n - 1 + \frac{\kappa_n(u) - u}{\theta_{n-1}(u)} \right) + a + 3 - u - \theta_{n-1}(u) + \frac{2u}{\theta_{n-1}(u)} \right|,$$

$$\leq t_> \max_{u \in (t, t_>)} \left| 2 \left( n - 1 + \frac{\kappa_n(u) - u}{\theta_{n-1}(u)} \right) + a + 3 + u + |\theta_{n-1}(u)| + \frac{2u}{|\theta_{n-1}(u)|} \right|$$

$$= O(n^{-1}), \quad \text{as } n \to \infty.$$
Thus the limit shown in (5.10) exists. Turning our attention to (5.12) we can use (3.1) to compute that
\[
\kappa_{n+1} - \kappa_n = -2\kappa_n - b_n\theta_n,
\]
\[
= -2\kappa_n - (2n + a + 3 - t - \theta_n)\theta_n,
\]
\[
= -2\theta_n\left(n + \frac{\kappa_n - t}{\theta_n}\right) - 2t - (a + 3)\theta_n + \theta_n(\theta_n + t),
\]
and therefore
\[
[k_{n+1}(t) - \kappa_n(t)][t = s/4n] = O(n^{-1}), \quad \text{as } n \to \infty.
\]
Now in this case the quantity we require is
\[
k_{n+1}(t)|_{t = s/4(n+1)} - \kappa_n(t)|_{t = s/4n} = k_{n+1}(t)|_{t = s/4(n+1)} - k_{n+1}(t)|_{t = s/4n} + [k_{n+1}(t) - \kappa_n(t)][t = s/4n],
\]
and therefore we need to bound the difference of first two terms on the right-hand side. Let us denote \(t_\prec = s/4(n+1)\). Again because \(\kappa_n(t)\) is continuously differentiable with derivative (3.22) we have
\[
|\kappa_{n+1}(t_\prec) - \kappa_{n+1}(t)|
\leq (t - t_\prec) \max_{u \in (t_\prec, t)} |\dot{\kappa}_{n+1}(u)|,
\]
\[
\leq \frac{s}{4n(n+1)} \max_{u \in (t_\prec, t)} |\dot{\kappa}_{n+1}(u)|,
\]
\[
\leq \frac{s}{4n(n+1)} \max_{u \in (t_\prec, t)} u^{-1}[\kappa_{n+1}(u) - a_n^2(\theta_{n+1}(u) - \theta_n(u))],
\]
\[
\leq \frac{1}{n} \max_{u \in (t_\prec, t)} \left(|\kappa_{n+1}(u)| + (2n + a + 4 + |\eta(u)|)u\right.
\]
\[
+ 2(n + a + 3)|\theta_{n+1}(u)|\left|n + 1 + \frac{\kappa_{n+1}(u) - u}{\theta_{n+1}(u)}\right|
\]
\[
= O(n^{-1}), \quad \text{as } n \to \infty.
\]
In the last two steps we have used (5.13) and the subsequent estimates. Thus the limit in (5.12) follows. The fact that the limit of \(\kappa_n(t)\) under the hard edge scaling is related to the limit given in (5.10) follows from the relation (3.2). The limit (5.11) is a consequence of the limits in the primary variables. \(\square\)

In addition the following combinations of variables possess scaling limits which will subsequently be useful.
Corollary 5.2. For bounded \( s \in \mathbb{R}_+ \) the following limits as \( n \to \infty \) exist
\[
\begin{align*}
(5.14) & \quad \lim_{t = s/4n \to \infty} \frac{2\kappa_n(t) + \theta_n(t)b_n(t)}{\theta_n(t)} \to s\frac{\dot{\mu}}{\mu}, \\
(5.15) & \quad \lim_{t = s/4n \to \infty} \left( n + \frac{\kappa_n(t) - t}{\theta_n(t)} \right) \to C(s), \\
(5.16) & \quad \lim_{t = s/4n \to \infty} \left( n + a + 2 + \frac{\kappa_n(t) + t}{\theta_{n-1}(t)} \right) \to -C(s), \\
(5.17) & \quad \lim_{t = s/4n \to \infty} n \left( n + \frac{\kappa_n(t) - t}{\theta_n(t)} + n + a + 2 + \frac{\kappa_n(t) + t}{\theta_{n-1}(t)} \right) \to \xi(s).
\end{align*}
\]

Proof. The limit in (5.15) is a consequence of (5.4) in a previous corollary. The scaling limit of (5.16) follows from that of (5.15) and the identity (3.11). The limit (5.17) can be derived from (3.12) and as a result one can deduce that
\[
n + \frac{\kappa_n - t}{\theta_n} + n + a + 2 + \frac{\kappa_n + t}{\theta_{n-1}}
\]
is of order \( O(n^{-1}) \).

We note other relations amongst the scaling limit functions, namely
\[
(5.18) \quad 2\mu C(s) = -[(a + 2)\mu + 2s] - \left\{ [(a + 2)\mu + 2s]^2 + 4\mu(\mu + s)\nu - \mu(\mu + s)^2 \right\}^{1/2},
\]
and
\[
(5.19) \quad \xi(s) = -s\frac{C(a + 1)}{\mu + s},
\]
and
\[
(5.20) \quad 2C + a + 3 = s\frac{\dot{\mu} - 2}{\mu}.
\]

Proposition 5.2. The scaled variables \( \mu(s), \nu(s) \) are characterised by solutions to the PIII system with parameters \( v_1 = a + 2, v_2 = a - 2 \). In particular
\[
(5.21) \quad \nu(s) = -\sigma_{III}(s) + \frac{1}{4}s - a - 2,
\]
where \( \sigma_{III}(s) \) satisfies the Jimbo-Miwa-Okamoto \( \sigma \)-form for PIII with above parameters. The boundary conditions to uniquely specify the solution \( \nu(s) \) are
\[
(5.22) \quad \nu(s) = \frac{a}{s-0} \frac{a}{a + 2} \frac{s}{4} - \frac{a}{8(a + 3)(a + 2)^2(a + 1)} s^2 + O(s^3)
\]
\[
- \frac{2}{4^{a+3}(a + 2)(a + 1)\Gamma(a + 3)\Gamma(a + 4)} s^{a+3} (1 + O(s)) + O(s^{2a+6}),
\]
assuming \( a \notin \mathbb{Z}_{\geq 0} \) and \( |\arg(s)| < \pi \).
Proof. Introducing the scaling ansatze (5.10) and (5.11) formally into the differential equation (3.34) under the scaling (5.1) we find that the highest order nontrivial relation

\[ \mu(s) + s = 4s\dot{\nu}(s), \]

at order \( n^{-1} \). Proceeding in the same manner with the differential equation (3.35) we find the highest order relation is

\[ s\dot{\mu}(s) = \mu + \left\{ [(a + 2)\mu + 2s]^2 + 4\mu(\mu + s)\nu - \mu(\mu + s)^2 \right\}^{1/2}, \]

which occurs at order \( n^{-1} \) as well. Eliminating \( \mu(s) \) using (5.23) we find that (5.24) yields

\[ s^2(\ddot{\nu})^2 - (a + 2)^2(\dot{\nu})^2 + \dot{\nu}(4\dot{\nu} - 1)(s\dot{\nu} - \nu) + \frac{1}{2}a(a + 2)\dot{\nu} - \frac{1}{16}a^2 = 0, \]

which is almost the Jimbo-Miwa-Okamoto \( \sigma \)-form for \( P_{III} \) [28]. The boundary conditions follow from the application of the scaling limit (5.11) to the expansion about \( t = 0 \), Equation (3.29). \( \square \)

In addition we find the scaling behaviour of the Hankel determinants and polynomial evaluations to be given by the following propositions.

**Proposition 5.3.** As \( n \to \infty \) under the hard edge scaling the Hankel determinants scale as

\[ \Delta_n(t)|_{t = s/4n} \sim \frac{1}{n!} \ldots (n - 1)!\Gamma(a + 3) \ldots \Gamma(n + a + 2) \exp \left( \int_0^s \frac{du}{u} \nu(u) \right), \]

and the monic polynomials evaluated at \( x = 0 \) scale as

\[ \pi_n(0; t)|_{t = s/4n} \sim (-1)^n (a + 3)_n \exp \left( \int_0^s \frac{du}{u} C(u) \right). \]

**Proof.** The relation (5.26) arises from integrating (3.27) with respect to \( t \) and then employing the scaling form for the integrand as given by (5.11). The second relation is derived by integrating (3.39) and using (5.15) for the scaling of the resulting integrand. The factorial and Pochhammer prefactors arise from the normalisations at \( t = 0 \). \( \square \)

We find that the polynomial ratios have well defined scaling behaviour rather than the polynomial themselves.

**Proposition 5.4.** The polynomial ratios \( Q_n(x; t), R_n(x; t) \) scale as

\[ \lim_{n \to \infty} Q_n(x; t)|_{x = -z/4n, t = s/4n} = q(z; s), \]

\[ \lim_{n \to \infty} nR_n(x; t)|_{x = -z/4n, t = s/4n} = p(z; s), \]

where \( q(z; s), p(z; s) \) are entire functions of \( z \).
Proof. We start with the product form of the scaled polynomial

\[(5.30) \quad Q_n\left(-\frac{z}{4n}; \frac{s}{4n}\right) = \prod_{j=1}^{n} \left(1 + \frac{z}{4nx_{j,n}(s/4n)}\right),\]

and seek bounds for the logarithm of the ratio of the \(n\)-th scaled polynomial to the \((n-1)\)-st. When \(0 < s, z < \infty\) we can write the general bound as the sum of four contributions

\[(5.31) \quad \left| \log \frac{Q_n\left(-\frac{z}{4n}; \frac{s}{4n}\right)}{Q_{n-1}\left(-\frac{z}{4(n-1)}; \frac{s}{4(n-1)}\right)} \right| \leq \left| \log \left(1 + \frac{z}{4nx_{n,n}(t)}\right) \right| \]

\[+ \sum_{j=1}^{n-1} \left\{ \left| \log \left(1 + \frac{z}{4nx_{j,n}(t)}\right) - \log \left(1 + \frac{z}{4(n-1)x_{j,n}(t)}\right) \right| \right. \]

\[+ \left| \log \left(1 + \frac{z}{4(n-1)x_{j,n}(t)}\right) - \log \left(1 + \frac{z}{4(n-1)x_{j,n}(t_\rangle)}\right) \right| \]

\[+ \left| \log \left(1 + \frac{z}{4(n-1)x_{j,n}(t_\rangle)}\right) - \log \left(1 + \frac{z}{4(n-1)x_{j,n-1}(t_\rangle)}\right) \right| \} , \]

where \(t = s/4n\) and \(t_\rangle = s/(4(n-1))\). Using the inequality

\[\log \frac{1 + A}{1 + B} < A - B\]

for \(A > B > 0\) we can find a simpler bound

\[(5.32) \quad \text{LHS of (5.31)} \leq \frac{z}{4nx_{n,n}(t)} \]

\[+ \sum_{j=1}^{n-1} \left\{ \left| \frac{z}{4nx_{j,n}(t)} - \frac{z}{4(n-1)x_{j,n}(t)} \right| \right. \]

\[+ \left| \frac{z}{4(n-1)x_{j,n}(t)} - \frac{z}{4(n-1)x_{j,n}(t_\rangle)} \right| \]

\[+ \left| \frac{z}{4(n-1)x_{j,n}(t_\rangle)} - \frac{z}{4(n-1)x_{j,n-1}(t_\rangle)} \right| \} . \]

Considering the first sum of (5.32) we see that this is

\[\sum_{j=1}^{n-1} \left| \frac{z}{4nx_{j,n}(t)} - \frac{z}{4(n-1)x_{j,n}(t)} \right| = \frac{z}{4(n-1)} \sum_{j=1}^{n-1} \frac{1}{x_{j,n}(t)} ,\]

\[< \frac{z}{4(n-1)} \sum_{j=1}^{n} \frac{1}{x_{j,n}(t)} ,\]

\[< \frac{z}{4(n-1)} \frac{n}{2} = zO(n^{-1}) , \quad \text{as } n \to \infty ,\]
for all bounded \( s \). The second term of (5.32) is
\[
\sum_{j=1}^{n-1} \frac{z}{4(n-1)x_{j,n}(t)} - \frac{z}{4(n-1)x_{j,n}(t_\succ)} = \frac{z}{4(n-1)} \sum_{j=1}^{n-1} \left| \frac{1}{x_{j,n}(t)} - \frac{1}{x_{j,n}(t_\succ)} \right|.
\]

The summand appearing here can be bounded in the following way
\[
\left| \frac{1}{x_{j,n}(t)} - \frac{1}{x_{j,n}(t_\succ)} \right| = \frac{|x_{j,n}(t_\succ) - x_{j,n}(t)|}{x_{j,n}(t_\succ)x_{j,n}(t)},
\]
(5.33)
\[
\leq \frac{1}{x_{j,n}(t_\succ)x_{j,n}(t)} (t_\succ - t) \max_{u \in (t,t_\succ)} |\hat{x}_{j,n}(u)|.
\]

Now from (2.96) we note that
\[
|\hat{x}_{j,n}(u)| = u^{-1} \frac{\theta_n(u) + u}{-\theta_n(u) + x_{j,n}(u)} x_{j,n}(u),
\]
and so furnishes a bound on (5.33)
\[
\left| \frac{1}{x_{j,n}(t)} - \frac{1}{x_{j,n}(t_\succ)} \right| \leq \frac{1}{x_{j,n}(t_\succ)x_{j,n}(t)} \frac{4n}{4n(n-1)} \frac{4n}{4n(n-1)} \frac{4n}{4n(n-1)} \frac{1}{x_{j,n}(t_\succ)} \frac{\theta_n(u) + u}{-\theta_n(u) + x_{j,n}(u)} x_{j,n}(u),
\]
\[
\leq \frac{1}{n-1} \frac{1}{x_{j,n}(t_\succ)x_{j,n}(t)} x_{j,n}(t) \max_{u \in (t,t_\succ)} \frac{\theta_n(u) + u}{-\theta_n(u) + x_{j,n}(u)} x_{j,n}(u),
\]
\[
\leq \frac{1}{n-1} \frac{1}{x_{j,n}(t_\succ)} \max_{u \in (t,t_\succ)} \frac{\theta_n(u) + u}{-\theta_n(u) + x_{j,n}(u)}.
\]

This means that
\[
\sum_{j=1}^{n-1} \left| \frac{1}{x_{j,n}(t)} - \frac{1}{x_{j,n}(t_\succ)} \right| \leq \frac{1}{n-1} \sum_{j=1}^{n-1} \max_{u \in (t,t_\succ)} \frac{\theta_n(u) + u}{-\theta_n(u) + x_{j,n}(u)} \frac{1}{x_{j,n}(t_\succ)},
\]
\[
\leq \frac{1}{n-1} \sum_{j=1}^{n-1} \max_{u \in (t,t_\succ)} \frac{\theta_n(u) + u}{-\theta_n(u) + x_{j,n}(u)} \sum_{j=1}^{n-1} \frac{1}{x_{j,n}(t_\succ)},
\]
\[
\leq \frac{1}{n-1} \sum_{j=1}^{n} \max_{u \in (t,t_\succ)} \frac{\theta_n(u) + u}{-\theta_n(u) + x_{j,n}(u)} \sum_{j=1}^{n} \frac{1}{x_{j,n}(t_\succ)},
\]
\[
\leq \frac{n}{2(n-1)} \sum_{j=1}^{n} \max_{u \in (t,t_\succ)} \frac{\theta_n(u) + u}{-\theta_n(u) + x_{j,n}(u)},
\]
\[
\leq \frac{n}{2(n-1)} \max_{u \in (t,t_\succ)} \sum_{j=1}^{n} \frac{\theta_n(u) + u}{-\theta_n(u) + x_{j,n}(u)},
\]
\[
\leq \frac{n}{2(n-1)} \max_{u \in (t,t_\succ)} \frac{n + \kappa_n(u) - u}{\theta_n(u)}.
\]

where we have used (2.94) in the last step. The total contribution of the second term is therefore bounded by
\[
\frac{zn}{8(n-1)^2} O(1) = zO(n^{-1}).
\]
The third sum in (5.32) is
\[
\frac{z}{4(n-1)} \sum_{j=1}^{n-1} \left| \frac{1}{x_{j,n}(t_\tau)} - \frac{1}{x_{j,n-1}(t_\tau)} \right| = \frac{z}{4(n-1)} \sum_{j=1}^{n-1} \left( \frac{1}{x_{j,n}(t_\tau)} - \frac{1}{x_{j,n-1}(t_\tau)} \right),
\]
\[
< \frac{z}{4(n-1)} \left( \sum_{j=1}^{n} \frac{1}{x_{j,n}(t_\tau)} - \sum_{j=1}^{n-1} \frac{1}{x_{j,n-1}(t_\tau)} \right).
\]

From the identity (3.48) and the scaling of the variables involved as given in (5.17) we conclude that the contribution of this sum is bounded by
\[
\frac{z}{4(n-1)} O(1) = zO(n^{-1}).
\]

Finally we note that the isolated term in (5.32) is of order \(zO(n^{-2})\) as a leading estimate of the largest zero \(x_{n,n}\) is of order \(O(n)\). This establishes that \(Q_n\) under the hard edge scaling of the independent variables converges to a limit as \(n \to \infty\), for all real, positive and bounded \(z, s\). \(\square\)

The spectral and deformation derivatives of the \(Q_n, R_n\) system scale to the corresponding derivatives of the \(q, p\) system as in the following result.

**Proposition 5.5.** Specify scaled quantities as in Proposition 5.1, Corollary 5.2 and Proposition 5.2. The spectral derivatives of the \(q, p\) system are
\[
(s - z)z \partial_z q = -zCq - (\mu + z)p,
\]
\[
(s - z)z \partial_z p = -z \left[ \xi + \frac{1}{4}(z - s) \right] q + [-2s + z(C + a + 2)]p,
\]
and their deformation derivatives are
\[
(s - z)s \partial_s q = zCq + (\mu + s)p,
\]
\[
(s - z)s \partial_s p = z\xi q - [s(2C + a) - zC]p.
\]
The boundary conditions satisfied by the solutions \(q(z; s)\) and \(p(z; s)\) of the above system on the domain \(s \geq z \geq 0\) along \(z = 0\) are
\[
q(0; s) = 1,
\]
\[
p(0; s) = 0,
\]
for all \(s > 0\).

**Proof.** The first spectral derivative (5.34) follows from the scaling of (3.53) and employing (5.10), (5.12) and (5.15). The second member (5.35) is derived from the scaling of (3.54) and using (5.15) and (5.17). The first deformation derivative (5.36) follows from the scaling of (3.55) and utilising (5.10), (5.12) and (5.15). The
second deformation derivative (5.37) arises from the scaling of (3.56), employing (5.16) and (5.17) and noting that

\[
(\kappa_n + t) \frac{\theta_{n-1} + t}{\theta_{n-1}} - (\kappa_n - t) \frac{\theta_n + t}{\theta_n} \bigg|_{t = s/4n, n \to \infty} \sim -\frac{s(2C + a)}{4n}.
\]

The boundary conditions (5.38) and (5.39) follow from the definitions (3.41) and (3.52) respectively and the scalings in Proposition 5.4.

**Remark 5.1.** Both the spectral derivative (3.43) and the deformation derivative (3.45) scale to

\[
(\mu + s)z \partial_z q + (\mu + z)s \partial_s q + zC(s)q = 0,
\]

and this mixed derivative equation can be easily found from Proposition 5.5 by eliminating the variable \( p \) between (5.34) and (5.36), i.e. \( (\mu + s) \) times (5.34) plus \( (\mu + z) \) times (5.36). The three-term recurrence relation (3.42) scales to

\[
z^2 \partial_z^2 q + 2sz \partial_z q + s^2 \partial_s^2 q + s \frac{\dot{\mu} - 2}{\mu} [z \partial_z q + s \partial_s q] - \frac{1}{4} z q = 0.
\]

This can also be recovered from the equations of Proposition 5.5. If we eliminate \( q \) between (5.34) and (5.36) by adding them we find that

\[
p = z \partial_z q + s \partial_s q.
\]

Employing this we find that (5.41) is equivalent to

\[
z \partial_z p + s \partial_s p + (-1 + s \frac{\dot{\mu} - 2}{\mu}) p - \frac{1}{4} z q = 0,
\]

which can be found by adding (5.35) and (5.37) and noting the relation (5.20).

**Remark 5.2.** In addition to the boundary conditions at \( z = 0 \) given by (5.38) and (5.39) there are also relations along \( s = z \)

\[
\frac{d}{ds} q(s; s) = -\frac{C}{\mu + s} q(s; s),
\]

and further

\[
\frac{p(s; s)}{q(s; s)} = \frac{\xi}{C + a} = -\frac{sC}{\mu + s}.
\]

However these are a consequence of the spectral and deformation derivatives and so do not constitute independent boundary conditions.

**Remark 5.3.** The compatibility of the two sets of derivatives, (5.34) and (5.35) on the one hand, and (5.36) and (5.37) on the other hand, affords a check on our results. We find that compatibility of (5.34) and (5.36) leads us to conclude that

\[
\xi = sC + \frac{1}{4} (\mu + s),
\]
and we also recover (5.20). Similar considerations applied to (5.35) and (5.37) imply that

\begin{equation}
\dot{s} \xi = -(2C + a + 2) \xi + \frac{1}{4} s(2C + a),
\end{equation}

and we get (5.46) again. Using (5.19) to eliminate \( \mu \) we arrive at a coupled pair of first order ordinary differential equations

\begin{align}
\dot{s} \dot{C} &= \xi + \frac{s}{4\xi} C(C + a), \\
\dot{s} \xi &= -(2C + a + 2) \xi + \frac{1}{4} s(2C + a).
\end{align}

Using the latter equation to eliminate \( C \) we obtain a second order ordinary differential equation for \( \xi \), which by means of the transformation

\begin{equation}
\xi(s) = \frac{1}{4} sy(s) - \frac{1}{y(s) - 1},
\end{equation}

is transformed into the standard equation for the fifth Painlevé transcendent. This is a degenerate case of \( P_V \) which reduces to the third Painlevé transcendent because the parameters are \( \alpha_V = 9/2, \beta_V = -a^2/2, \gamma_V = 1/2, \delta_V = 0 \). Making an independent variable transformation \( s \mapsto 2s \) so that \( \gamma_V = 1 \) we determine the \( P_{III} \) parameters to be \( \alpha_{III} = 2(2 - a), \beta_{III} = 2(a + 3), \gamma_{III} = 1, \delta_{III} = -1 \) which is consistent with those in Proposition 5.2.

The matrix form of the spectral derivatives (5.34,5.35) and deformation derivatives (5.36,5.37) yield the Lax pairs

\begin{align}
\partial_z \Psi &= \left\{ A_{\infty} + \frac{A_s}{z - s} + \frac{A_0}{z} \right\} \Psi, \\
\partial_s \Psi &= \left\{ B - \frac{A_s}{z - s} \right\} \Psi,
\end{align}

in the matrix variable

\begin{equation}
\Psi(z; s) = \begin{pmatrix} q(z; s) \\ p(z; s) \end{pmatrix}.
\end{equation}

The system has two regular singularities at \( z = 0, s \) and an irregular one at \( z = \infty \) with a Poincaré index of 1. This system is essentially equivalent to the isomonodromic system of the fifth Painlevé equation but is the degenerate case. The
residue matrices are

\[(5.54)\]
\[A_0 = \begin{pmatrix} 0 & -\frac{\mu(s)}{s} \\ 0 & -2 \end{pmatrix},\]

\[(5.55)\]
\[A_s = \begin{pmatrix} C(s) & \frac{\mu(s) + s}{s} \\ \xi(s) & -C(s) - a \end{pmatrix},\]

\[(5.56)\]
\[A_\infty = \begin{pmatrix} 0 & 0 \\ \frac{1}{4} & 0 \end{pmatrix},\]

\[(5.57)\]
\[B = \frac{1}{s} \begin{pmatrix} -C(s) & 0 \\ -\xi(s) & -C(s) \end{pmatrix}.\]

Local convergent expansions about the regular singularities take the form for \(z = 0\)

\[(5.58)\]
\[q(z; s) = \sum_{m \geq 0} r_m^0 (s) z^{r_0^m},\]

\[(5.59)\]
\[p(z; s) = \sum_{m \geq 0} u_m^0 (s) z^{u_0^m},\]

for \(|z| < s\) and the initial relations amongst the coefficients are found to be

\[(5.60)\]
\[s\chi_0 r_0^0 = -\mu u_0^0, \quad s\chi_0 u_0^0 = -2 su_0^0.\]

This implies that \(\chi_0 = -2\) or \(\chi_0 = 0\) and \(u_0^0 = 0\). The latter case applies here as both \(q, p\) are analytic at \(z = 0\), and in addition we also require \(p = 0\) on \(z = 0\). In addition \(r_0^0 = 1\). The recurrence relations for general \(m\) are

\[(5.61)\]
\[s(m + 2) u_m^0 = (C + a + m + 1) u_{m-1}^0 + \left(\frac{1}{4} s - \xi\right) r_{m-1}^0 - \frac{1}{4} r_{m-2}^0,\]

\[(5.62)\]
\[s m r_m^0 = -\mu u_m^0 + (m - 1 - C) r_{m-1}^0 - u_{m-1}^0.\]

For \(z = s\) we have the convergent expansion

\[(5.63)\]
\[q(z; s) = \sum_{m \geq 0} r_m^s (s) (s - z)^{r_s^m},\]

\[(5.64)\]
\[p(z; s) = \sum_{m \geq 0} u_m^s (s) (s - z)^{u_s^m},\]

for \(|z - s| < s\) and where in this case the initial relations are

\[(5.65)\]
\[s(-\chi_s + C) r_0^s = - (\mu + s) u_0^s, \quad (C + a + \chi_s) u_0^s = \xi r_0^s.\]

Combining these we get a relation which is identical to the second equality in (5.45) only if \(\chi_s = 0, -a\). The former case is the one we must choose as \(q, p\) are well-defined.
and finite on $z = s$. For general $m$ the recurrence relations are
\begin{equation}
(5.66)
s(C - m)r^s_m + (\mu + s)u^s_m = (C - m + 1)r^s_{m-1} + u^s_{m-1},
\end{equation}
and
\begin{equation}
(5.67)
-\xi r^s_m + s(C + a + m)u^s_m = - \left( \frac{1}{4} s + \xi \right) r^s_{m-1} + (C + a + m + 1)u^s_{m-1} + \frac{1}{4} r^s_{m-2}.
\end{equation}

This system has a unique solution only if $s^2(a + m) \neq 0$ for $s > 0$ and $m \geq 1$ which in turn means that $a \neq -N$. We also note that the two sets of coefficients are related by
\begin{equation}
(5.68)
r^s_m(s) = (-1)^m \sum_{n=0}^{\infty} \binom{m+n}{m}s^nr^0_{m+n}(s),
\end{equation}
and an identical relation for $u^s_m(s)$.

**Proposition 5.6.** The determinant $D_n(x, x)$ scales as
\begin{equation}
(5.69)
D_n(x, x)|_{x = z/4n, t = s/4n} \sim -4\Delta_n \pi_n(0)\pi_{n+1}(0) [q\partial_z p - p\partial_z q].
\end{equation}

**Proof.** We can employ the polynomials $Q_n, R_n$ in the evaluation (2.45) and find
\begin{equation}
D_n(x, x) = \Delta_n \pi_n(0)\pi_{n+1}(0) [Q^0_{n+1}R'_n - R^0_n Q'_{n+1}].
\end{equation}
Applying the scaling of Proposition 5.4 to this expression we arrive at (5.69). \square

**Proposition 5.7.** The distribution $A_n,a(y)$ scales to
\begin{equation}
(5.70)
\frac{1}{4n} A_n,a(y)|_{y = z/4n} \sim_{n \to \infty} A_a(z).
\end{equation}

**Proof.** This is the only possible scaling consistent with the scaling of the independent variable to the hard edge, $y = z/4n$. \square

**Proposition 5.8.** The distribution of the first eigenvalue spacing at the hard edge is given by
\begin{equation}
(5.71)
A_a(z) = \frac{z^2}{4^{2a+3}\Gamma(a+1)\Gamma(a+2)\Gamma^2(a+3)}
\times \int_{-\infty}^{\infty} ds s^a(s-z)^a \exp \left( -\frac{s}{4} + \int_0^s \frac{dv}{v} [\nu(v) + 2C(v)] \right) [q\partial_z p - p\partial_z q].
\end{equation}

**Proof.** We apply the scaling (5.70) to Eq. (1.31) and utilise the previous relation for the scaling of the integrand (5.69). For the first three factors on the right-hand side of (5.69) we can use the scaling results of (5.26) and (5.27), yielding the above integral representation. \square

We note that the factor of the integrand of (5.71) can be written as
\begin{equation}
(5.72)
q\partial_z p - p\partial_z q = \frac{1}{4} q^2 - \frac{2}{z} qp + \frac{\mu}{sz} p^2 - \frac{1}{\xi}(\xi q - Cp) \frac{\xi q - (C + a)p}{s - z}.
\end{equation}
The corresponding distribution \( A^\pm(z) \) defined in (1.35) and (1.36) for the special cases \( a = \pm 1/2 \) is given by

\[
(5.73) \quad A^\pm(z) = \frac{z^2}{4^{2a+2} \Gamma(a+1) \Gamma(a+2) \Gamma^2(a+3)} \times \int_z^\infty ds \, s^{2a+1}(s-z)^{2a+1}(2s-z)^2 \exp \left( -\frac{s^2}{4} + \int_0^s \frac{dv}{v} [\nu(v) + 2C(v)] \right) \\
\times [q \partial_{z} p - p \partial_{z} q] \bigg|_{s \to z(2s-z)}.
\]

5.2. Special Case \( a \in \mathbb{Z}_{\geq 0} \).

**Proposition 5.9.** In the special case \( a \in \mathbb{Z}_{\geq 0} \) we have

\[
(5.74) \quad \Delta_n(t) |_{t=s/4} \sim \frac{c_{n+1,n+1+a}}{(n+a)!} (-1)^{\lfloor a/2 \rfloor} (2n)^a s^{-a} \det [I_{j+2-k}(\sqrt{s})]_{j,k=1,\ldots,a},
\]

and

\[
(5.75) \quad \nu(s) = s \frac{d}{ds} \log s^{-a} \det [I_{j+2-k}(\sqrt{s})]_{j,k=1,\ldots,a},
\]

whilst for \( a = 0 \) we find \( \nu(s) = 0 \).

*Proof.* The first relation (5.74) follows from an application of the Hilb type asymptotic formula (5.2) to (4.1) and the second follows by using this result in (3.27). \( \square \)

Another consequence of the Hilb formula is the scaling of the orthogonal polynomial ratio as given by (4.8).

**Proposition 5.10.** The scaled orthogonal polynomial ratio \( q(z;s) \) is

\[
(5.76) \quad q(z;s) = z^{-3/2} \left( \frac{s}{s-z} \right)^a \frac{\det \left[ \begin{array}{c} [z/s]^{k/2} I_{3-k}(\sqrt{s}) \\ [I_{j+2-k}(\sqrt{s})]_{j=1,\ldots,a+1} \\ [1/8\sqrt{s}, 1/2s, 1/s^{3/2}, 0, \ldots, 0] \end{array} \right]}{\det \left[ [I_{j+2-k}(\sqrt{s})]_{j=1,\ldots,a} \right]}, \quad a \geq 1
\]

and for \( a = 0 \) is

\[
(5.77) \quad q(z;s) = \frac{8}{z} I_2(\sqrt{z}).
\]

Finally the eigenvalue spacing distribution (4.16) takes the following form in the scaling limit.
Proposition 5.11. The distribution of the first eigenvalue spacing at the hard edge for \( a \in \mathbb{Z}_{>0} \) is

\[
A_a(z) = 2^{-4} z^{1/2} \int_z^\infty ds \frac{s}{s-1/2} a^{-s/4} e^{-s/4} \times \det \left[ I_{j+2-k}(\sqrt{s}) \right]_{k=1,\ldots,a+2}^{j=1,\ldots,a+2}
\times \left[ (s/z)^{2-k/2} I_{j+2-k}(\sqrt{s}) \right]_{j=1,2}^{k=1,\ldots,a+2}
\]

for \( a \geq 1 \) and for \( a = 0 \) is

\[
A_0(z) = \frac{1}{4} e^{-z/4} \left[ I_2^2(\sqrt{z}) - I_1(\sqrt{z}) I_3(\sqrt{z}) \right].
\]

Proof. We apply the Hilb asymptotic formula (5.2) to (4.16). As \( \frac{1}{2}(a+1)(a+2) + 1 + [a/2] \) is always even for \( a \in \mathbb{Z} \) we have (5.78). \( \square \)

6. Analytical Studies at the Hard Edge

In this section of our study we intend to develop the analytical and non-formal theory of the solutions to the defining ordinary and partial differential equations described in the previous sections. This is because we wish to compute precision numerical data characterising the distribution function of the first eigenvalue spacing at the hard edge \( A_a(z) \) for arbitrary parameter \( a \). For this purpose it is not sufficient to employ a single local expansion of the \( \sigma \)-function, about \( s = 0 \) say, because it has a finite convergence domain and one cannot use this to evaluate the \( s \)-integrals on the interval \([0, \infty)\). For this reason we construct a patchwork of overlapping local expansions including Taylor series expansions about regular points \( s_0 \) for positive and real values. A similar approach was undertaken by Prähofer and Spohn in their study [29] of the exact scaling functions for one-dimensional stationary KPZ growth.

6.1. The \( \sigma \)-function expansion about \( s = 0 \). The terms given in the expansion of the Painlevé III’ \( \sigma \)-function (5.22) are the minimum required to specify the full non-analytic Puiseux-type expansion of the particular solution for \( \nu(s) \) about \( s = 0 \) in the sector \( -\pi \leq \arg(s) < \pi \). This is the primary data specifying our particular solution and we need to use this in various ways in order to compute the distribution \( A_a(z) \).

Proposition 6.1 ([18],[21]). The Painlevé III’ \( \sigma \)-function \( \nu(s) \) has a Puiseux-type expansion about the fixed regular singular point \( s = 0 \) of the form

\[
\nu(s) = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} c_{k,j} s^{j+k a}, \quad a \in \mathbb{C}, \quad 0 \leq \text{Re}(a) < 1,
\]

with \( |\arg(s)| < \pi \) and is convergent in a finite domain \( s \in \{ z \in \mathbb{C} : |z| < R, |z^a| < R_a \} \).
The coefficients \( c_{k,j} \) are determined by recurrences which follow from the substitution of expansion (6.1) into the relation (5.25). We will assume that \( a \) is not rational for simplicity. Considering terms in the resulting equation with \( s^p \) and \( p \in \mathbb{Z}_{\geq 0} \) then the \( p = 0 \) case implies that if \( c_{0,0} = 0 \) then this choice fixes
\[
(6.2) \quad c_{0,1} = \frac{a}{4(a + 2)}.
\]
The \( p = 1 \) is automatically zero but for the \( p = 2 \) case one has the two options
\[
(6.3) \quad c_{0,2} = -\frac{a}{8(a + 3)(a + 2)^2(a + 1)} \quad \text{or} \quad 0.
\]
The former case applies here by comparison with (5.22). For \( p \geq 3 \) the recurrence is
\[
(6.4) \quad -\frac{1}{2} a(p + a + 1)(p - a - 3) c_{0,p} - \frac{2}{a + 2} \sum_{j=2}^{p-1} j(p - j) c_{0,j} c_{0,p+1-j} \]
\[
+ 8 c_{0,2} \sum_{j=2}^{p-1} (j - 1)(p - j) c_{0,j-1} c_{0,p+1-j} \]
\[
+ \sum_{j=3}^{p-1} (j(p + 2 - j)[(j - 1)(p + 1 - j) - (a + 2)^2] c_{0,j} c_{0,p+2-j} \]
\[
+ 4 \sum_{j=3}^{p-1} \sum_{m=1}^{p-1} j m(p + 1 - j - m) c_{0,j} c_{0,m} c_{0,p+2-j-m} = 0,
\]
which allows for \( c_{0,p} \) to be recursively found, valid for \( a \notin \mathbb{Z} \). Terms with \( s^{-2+qa} \) for \( q = 2k \geq 2 \) imply that \( c_{k,0} = 0 \) given that \( c_{0,0} = 0 \) and \( a \neq 3/(k - 1), -1/(k + 1) \). Consequently all terms with \( s^{-1+qa} \) vanish also. The generic recurrence relation following from examining the \( s^{p+qa} \) term is
\[
(6.5) \quad \frac{1}{2} a(a + 2)(p + 1 + qa) c_{q,p+1} = \sum_{k=0}^{p+1} \sum_{j=0}^{q} (j + ka)[p - j + (q - k)a] c_{k,j} c_{q-k,p+1-j} \]
\[
- \sum_{k=0}^{q} \sum_{j=0}^{p+2-j} (j + ka)[p + 2 - j + (q - k)a][j - 1 + ka][p + 1 - j + (q - k)a] - (a + 2)^2] c_{k,j} c_{q-k,p+2-j} \]
\[
- 4 \sum_{k=0}^{q} \sum_{l=0}^{p+2-j} \sum_{m=0}^{p+2-j} (j + ka)(m + la) [p + 1 - j - m + (q - k - l)a] c_{k,j} c_{l,m} c_{q-k-l,p+2-j-m},
\]
for \( q, p \geq 0 \). The convention is taken that sums with upper limits less than their lower limits are zero, or equivalently coefficients with negative indices are zero. Contrary to appearances the highest coefficient in (6.5) will turn out to be \( c_{q,p} \) as \( c_{q,p+2} \) occurs with \( c_{0,0} \) as a factor and \( c_{q,p+1} \) has a factor of
\[
(6.6) \quad \frac{1}{2} a(a + 2) + c_{0,0} - 2(a + 2)^2 c_{0,1} - 8 c_{0,0} c_{0,1},
\]
which is zero as a consequence of $c_{0,0} = 0$ and (6.2). We find that the coefficient $c_{q,p}$ occurs as a linear term and has a factor of

$$(6.7) \quad (p - 1 + qa)c_{0,1}(4c_{0,1} - 1) + 4(p + qa)(p - 1 + qa - (a + 2)^2)c_{0,2} - 16(p + qa)c_{0,0}c_{0,2}. $$

With the above evaluations (6.2) and (6.3) this is

$$(6.8) \quad -\frac{a(p + 1 + (q + 1)a)(p - 3 + (q - 1)a)}{2(a + 3)(a + 2)^2(a + 1)}. $$

This vanishes when $q = 1, p = 3$ and we find that $c_{1,3}$ is undetermined and therefore is a free parameter. With these choices a triangular subset of the coefficients vanish

$$(6.9) \quad c_{k,j} = 0, \quad j = 0, \ldots, 3k - 1 \text{ for } k = 1, \ldots, \infty, $$

so that the initial non-zero term in the $j$–sum is $c_{k,3k}$. All other terms are fixed by $c_{1,3}$ and $a$.

For the variable $\mu(s)$ we can deduce the following Puiseux-type expansion from the above work

$$(6.10) \quad \mu(s) \sim -\frac{2}{a + 2}s - \frac{a}{(a + 3)(a + 2)^2(a + 1)}s^2 - \frac{2}{4^{a + 2}(a + 2)(a + 1)^2(a + 3)^2} s^{a + 3}. $$

In regard to the quantity $2C(s)$ we require as much detail about this as for $\nu(s)$. This variable is a $\sigma$-function for an identical problem, where the fixed exponent 2 in the weight (1.32) is replaced by 3. If we define the expansion coefficients for this object

$$(6.11) \quad 2C(s) = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} a_{k,j}s^{j+ka}, \quad a \in \mathbb{C}, \quad 0 \leq \text{Re}(a) < 1, $$

then the defining recurrences for these using (5.20) are

$$(6.12) \quad a_{0,0} = 0, \quad a_{0,1} = 4(a + 2)(a + 1)c_{0,2}, $$

and for $j \geq 2$

$$(6.13) \quad a_{0,j} = 2(a + 2)\left[ - (j + 1)(j - 2 - a)c_{0,j+1} + \sum_{l=2}^{j} la_{0,j+1-l}c_{0,l} \right]. $$

For the case $k \geq 1$ we have $a_{k,j} = 0$ for $j = 0, \ldots, 3k - 1$ and the remaining non-zero terms are given by

$$(6.14) \quad a_{1,j} = 2(a + 2)\left[ - (j + 1 + a)(j - 2)c_{1,j+1} + \sum_{l=0}^{j} (l + a)a_{0,j+1-l}c_{1,l} + \sum_{l=2}^{j+1} la_{1,j+1-l}c_{0,l} \right], $$
for $k = 1$ and the general case $k \geq 2$ by

$$a_{k,j} = 2(a + 2)\left[ -(j + 1 + ka)(j - 2 + (k - 1)a)c_{k,j+1}
+ \sum_{l=0}^{j}(l + ka)a_{0,j+1-l}c_{k,l} + \sum_{l=2}^{j+1}la_{k,j+1-l}c_{0,l}
+ \sum_{m=1}^{k-1} \sum_{l=0}^{j+1}(l + ma)a_{k-m,j+1-l}c_{m,l}\right].$$

The first few terms are

$$2C(s) \sim -\frac{a}{2(a + 3)(a + 2)}s - \frac{a(a^2 - 5a - 18)}{8(a + 4)(a + 3)^2(a + 2)^2(a + 1)}s^2
+ \frac{1}{4a^2(a + 2)(a + 1)}\Gamma(a + 3)\Gamma(a + 4)s^{a+3}. $$

and

$$\xi(s) \sim \frac{a}{4(a + 3)}s - \frac{3a}{8(a + 4)(a + 3)^2(a + 2)}s^2
+ \frac{3}{4a^2(a + 3)(a + 2)(a + 1)}\Gamma^2(a + 4)s^{a+4}. $$

6.2. The $q,p$ expansion about $s = 0$. In this subsection we seek local expansions about $s = 0$ for the coefficients functions $r_m^0(s), u_m^0(s), r_m^s(s), u_m^s(s)$ appearing in (5.58), (5.59), (5.63) and (5.64). In parallel with the transcendent quantities these will have Puiseux-type expansions of the form

$$r_m^0(s) = \sum_{k,j \geq 0} r_{m,k,j}^0 s^{j+ka},$$

$$r_m^s(s) = \sum_{k,j \geq 0} r_{m,k,j}^s s^{j+ka},$$

with analogous expansions for the remaining two coefficients. This is immediately clear because the recurrence relations for these imply that they are polynomial functions of the variables $\mu(s), C(s), \xi(s)$. These recurrence relations imply the following ones for the $z = 0$ coefficients $r_{m,k,j}^0, u_{m,k,j}^0$

$$(a + m + 2)u_{m,k,j}^0 = \frac{1}{4}r_{m-1,k,0}^0, \quad mr_{m,k,0}^0 = u_{m,k,0}^0,$$

for $m \geq 1, k \geq 0$. For the general case $j \geq 1, k \geq 0$ we have

$$(m + 2)u_{m,k,j-1}^0 = (a + m + 1)u_{m-1,k,j}^0 + \frac{1}{2} \sum_{q=0}^{k} \sum_{p=0}^{j} a_{k-q,j-p}u_{m-1,q,p}^0
+ \frac{1}{4}r_{m-1,k,j-1}^0 - \sum_{q=0}^{k} \sum_{p=0}^{j}(j-p+(k-q)a) \left[c_{k-q,j-p}^0 + \frac{1}{2}a_{k-q,j-p}^0\right]r_{m-1,q,p}^0 - \frac{1}{4}r_{m-2,k,j}^0,$$
and

\[(6.22) \quad m r_{m,k,j}^0 = (m-1) r_{m-1,k,j}^0 - \frac{1}{2} \sum_{q=0}^{k} \sum_{p=0}^{j} a_{k-q,j-p} r_{m-1,q,p}^0 \\
+ u_{m,k,j}^0 - 4 \sum_{q=0}^{k} \sum_{p=0}^{j} [j - p + (k - q)a]c_{k-q,j-p}u_{m,q,p}^0 - u_{m-1,k,j}^0,\]

These equations can be solved for successive values of \(m\) starting with the \(m = 0\) values for all \(k, j\)

\[(6.23) \quad u_{0,k,j}^0 = 0, \quad r_{0,k,j}^0 = 0, \quad \text{except for } k = j = 0 \text{ where } r_{0,0,0}^0 = 1,\]

which follow from \(r_0^0(s) = 1, u_0^0(s) = 0\). The next few coefficients can be read off from

\[(6.24) \quad r_1^0(s) \sim \frac{1}{4(a+3)} + \frac{a}{16(a+4)(a+3)^2(a+2)} s \sum_{m=1}^{a} \frac{1}{128.4^m(a+3)(a+2)(a+1)\Gamma(a+5)\Gamma(a+4)} s^{a+3},\]

\[(6.25) \quad u_1^0(s) \sim \frac{1}{4(a+3)} + \frac{a}{8(a+4)(a+3)^2(a+2)} s \sum_{m=1}^{a} \frac{1}{128.4^m(a+3)(a+2)(a+1)\Gamma^2(a+4)} s^{a+3},\]

\[(6.26) \quad r_2^0(s) \sim \frac{1}{32(a+4)(a+3)} + \frac{a}{64(a+5)(a+4)(a+3)^2(a+2)} s \sum_{m=1}^{a} \frac{1}{512.4^m(a+3)(a+2)(a+1)\Gamma(a+6)\Gamma(a+4)} s^{a+3},\]

\[(6.27) \quad u_2^0(s) \sim \frac{1}{16(a+4)(a+3)} + \frac{3a}{64(a+5)(a+4)(a+3)^2(a+2)} s \sum_{m=1}^{a} \frac{1}{512.4^m(a+3)(a+2)(a+1)\Gamma(a+5)\Gamma(a+4)} s^{a+3}.\]

The analogous results for \(r^a_{m,k,j}, u^a_{m,k,j}\) are

\[(6.28) \quad (a + m + 2)u_{m,k,0}^a = \frac{1}{4} r_{m-1,k,0}^a, \quad m r_{m,k,0}^a = u_{m,k,0}^a.\]
for $m \geq 1, k \geq 0$. Again for the general case $j \geq 1, k \geq 0$ we have

\begin{equation}
(6.29) \quad (a + m)u_{m,k,j-1}^s + \frac{1}{2} \sum_{q=0}^{k-1} \sum_{p=0}^{j-1} a_{k-q,j-1-p}u_{m,q,p}^s
\end{equation}

\begin{align*}
&= (a + m + 1)u_{m-1,k,j}^s + \frac{1}{2} \sum_{q=0}^{k-1} \sum_{p=0}^{j-1} a_{k-q,j-1-p}u_{m-1,q,p}^s \\
&+ \sum_{q=0}^{k-1} \sum_{p=0}^{j-1} (j - 1 - p + (k - q)a) \left[ c_{k-q,j-1-p} + \frac{1}{2}a_{k-q,j-1-p} \right] r_{m,q,p}^s \\
&- \frac{1}{4} r_{m-1,k,j-1}^s - \sum_{q=0}^{k-1} \sum_{p=0}^{j-1} (j - p + (k - q)a) \left[ c_{k-q,j-1-p} + \frac{1}{2}a_{k-q,j-1-p} \right] r_{m-1,q,p}^s \\
&+ \frac{1}{4} r_{0,m-2,k,j}^s,
\end{align*}

and

\begin{equation}
(6.30) \quad mr_{m,k,j-1}^s - \frac{1}{2} \sum_{q=0}^{k-1} \sum_{p=0}^{j-1} a_{k-q,j-1-p}r_{m,q,p}^s
\end{equation}

\begin{align*}
&= (m - 1)r_{m-1,k,j}^s - \frac{1}{2} \sum_{q=0}^{k-1} \sum_{p=0}^{j-1} a_{k-q,j-1-p}r_{m-1,q,p}^s \\
&- u_{m-1,k,j}^s + \frac{1}{4} \sum_{q=0}^{k-1} \sum_{p=0}^{j-1} (j - p + (k - q)a) c_{k-q,j-1-p}u_{m,q,p}^s,
\end{align*}

Again these can be solved for successive values of $m$ starting with the initial values given by the relations

\begin{equation}
(6.31) \quad r_{0,k,j}^s = \sum_{n=0}^{j} r_{n,k,j-n}^0,
\end{equation}

along with an identical formula for $u_{0,k,j}^s$.

6.3. The $\sigma$-function expansion about $s = \infty$. The nature of the expansion of $\nu(s)$ about $s = \infty$ is rather different because this fixed singular point is irregular in the case of the third Painlevé transcendent.

**Proposition 6.2.** The formal asymptotic expansion of $\nu(s)$ about $s = \infty$ has the form

\begin{equation}
(6.32) \quad \nu(s) \sim \sum_{j=-1}^{\infty} d_j s^{-j/2},
\end{equation}

where $d_{-1} = \pm \frac{1}{2}a$.

**Proof.** We start with the general ansatz of

\begin{equation}
(6.33) \quad \nu(s) = s^{k\alpha} + O(s^{(k-1)\alpha}),
\end{equation}
where \( k \in \mathbb{N} \) and \( \alpha \in \mathbb{C} \) with \( 0 < \Re(\alpha) < 1 \). Using (5.25) we find that the only terms which can balance the \( \mathcal{O}(1) \) term are the \( \mathcal{O}(s^{2k\alpha-1}) \) and \( \mathcal{O}(s^{3k\alpha-2}) \) terms. If \( k\alpha < 1 \) then the former choice applies and we find \( k\alpha = 1/2 \). If we assume \( k\alpha > 1 \) then the latter case must be chosen but this leads to \( k\alpha = 2/3 \) in contradiction to the hypothesis. With the correct choice of \( k\alpha \) we find the above relation for leading coefficient, \( d_{-1} \). Considering the sub-leading terms we note that it is only possible for the terms of \( \mathcal{O}(s^{-\alpha}) \) to balance those of \( \mathcal{O}(s^{-1/2}) \) so that in fact \( \alpha = 1/2 \) and \( k = 1 \). This is entirely consistent with the expansion for the Painlevé III transcendent \( q_{\Pi}(t) \) about \( t = \infty \) as found in [15].

The choice of the sign of the leading coefficient is positive in our application. Consequently we find the first few terms of the asymptotic expansions of the \( \sigma \)-function

\[
\nu(s) = \left. \frac{1}{2} as^{1/2} - \frac{1}{4} a(a+4) + \frac{15}{16} a s^{-1/2} + \frac{15}{16} a^2 s^{-1} \right|_{s \to \infty} + \frac{15}{256} a(16a^2 - 7)s^{-3/2} + \mathcal{O}(s^{-2}),
\]

and the auxiliary variables

\[
\mu(s) + s = \left. as^{1/2} - \frac{15}{8} a s^{-1/2} - \frac{15}{4} a^2 s^{-1} \right|_{s \to \infty} - \frac{45}{128} a(16a^2 - 7)s^{-3/2} + \mathcal{O}(s^{-2}),
\]

and

\[
2C(s) + a = \left. \frac{5}{2} a s^{-1/2} + \frac{5}{2} a^2 s^{-1} + \frac{5}{16} a(8a^2 - 21)s^{-3/2} + \mathcal{O}(s^{-2}) \right|_{s \to \infty},
\]

and

\[
\xi(s) = \left. \frac{1}{4} as^{1/2} - \frac{35}{32} a s^{-1/2} - \frac{35}{16} a^2 s^{-1} + \mathcal{O}(s^{-3/2}) \right|_{s \to \infty}.
\]

The large \( s \)-regime also implies a simplification of spectral derivative (5.51) which becomes

\[
\partial_z \Psi = \left( -\frac{a}{2(z-s)} - \frac{1}{2(z-s)} - \frac{a}{2(z-s)} \right) \left\{ 1 + \mathcal{O}(s^{-1/2}) \right\} \Psi.
\]

Using the substitution \( \Psi \mapsto \exp(-a \ln(s-z) \| /2) \Psi \) this decouples and can be solved in terms of the modified Bessel functions. An application of the boundary condition (5.38) implies that

\[
q(z;s) \sim \left( \frac{s}{s-z} \right)^{a/2} \frac{8}{z} I_2(\sqrt{z}),
\]

\[
p(z;s) \sim \left( \frac{s}{s-z} \right)^{a/2} \frac{d}{dz} \left( \frac{8}{z} I_2(\sqrt{z}) \right).
\]
6.4. The \( \sigma \)-function expansion about a regular point. In this subsection we seek Taylor series expansions for the sigma function and derived variables about regular points \( s_0 \), taken to be positive and real without any loss of generality. Let us write

\[
\nu(s) = \sum_{j=0}^{\infty} d_j (s - s_0)^j,
\]

and using (5.25) we find the following recurrence relations for the coefficients \( d_j \)

\[
4s_0^2d_2 - \frac{1}{16}a^2 + \frac{1}{2}a(a + 2)d_1 + d_0d_1 - [s_0 + (a + 2)^2]d_1^2 + 4(s_0d_1 - d_0)d_1^2 = 0,
\]

and for \( n \geq 1 \), \( d_2 \neq 0 \)

\[
4s_0^2(n + 2)(n + 1)d_2d_{n+2} = -\frac{1}{2}a(a + 2)(n + 1)d_{n+1}
\]

\[
+ [s_0 + (a + 2)^2] \sum_{j=0}^{n} (j + 1)(n - j + 1)d_{j+1}d_{n-j+1}
\]

\[
- \sum_{j=0}^{n} (j + 1)(n - j - 1)d_{j+1}d_{n-j}
\]

\[
- \sum_{j=1}^{n-1} (j + 1)j(n - j + 1)(n - j)d_{j+1}d_{n-j+1}
\]

\[
- 2s_0 \sum_{j=0}^{n-1} (j + 2)(j + 1)(n - j + 1)(n - j)d_{j+2}d_{n-j+1}
\]

\[
- s_0^2 \sum_{j=1}^{n-1} (j + 2)(j + 1)(n - j + 2)(n - j + 1)d_{j+2}d_{n-j+2}
\]

\[
- 4 \sum_{i=0}^{n} \sum_{j=0}^{n-i} (i + 1)(j + 1)(n - i - j - 1)d_{i+1}d_{j+1}d_{n-i-j}
\]

\[
- 4s_0 \sum_{i=0}^{n} \sum_{j=0}^{n-i} (i + 1)(j + 1)(n - i - j + 1)d_{i+1}d_{j+1}d_{n-i-j+1}.
\]

These recurrences are solved subject to the initial values of

\[
d_0 = \sum_{k=0}^{\infty} \sum_{j=3k}^{\infty} c_{k,j}s_0^{j+k},
\]

\[
d_1 = \sum_{k=0}^{\infty} \sum_{j=3k}^{\infty} (j + ka)c_{k,j}s_0^{j-1+k},
\]
which in turn can be found from the solutions to the recurrences (6.4) and (6.5).
In addition we define

\[
\mu(s) = \sum_{j \geq 0} f_j(s - s_0)^j,
\]
\[
C(s) = \sum_{j \geq 0} g_j(s - s_0)^j,
\]
\[
\xi(s) = \sum_{j \geq 0} h_j(s - s_0)^j.
\]

The coefficients appearing here are computed using the recurrences

\[
f_0 = s_0(4d_1 - 1), \quad f_1 = 4d_1 - 1 + 8s_0d_2, \quad f_j = 4[jd_j + s_0(j + 1)d_{j+1}], \quad j \geq 2,
\]
and subject to \( f_0 \neq 0 \)

\[
2f_{0}g_0 = -2s_0 - (a + 3)f_0 + s_0f_1,
\]
\[
2f_{0}g_1 = -2 - (a + 2)f_1 + 2s_0f_2 - 2f_1g_0,
\]
\[
2f_{0}g_j = (j - a - 3)f_j + (j + 1)s_0f_{j+1} - 2\sum_{k=0}^{j-1} f_{j-k}g_k, \quad j \geq 2,
\]
and provided \( s_0 + f_0 \neq 0 \)

\[
(s_0 + f_0)h_0 = -s_0g_0(a + g_0),
\]
\[
(s_0 + f_0)h_1 = -as_0g_1 - 2s_0g_0g_1 - (a + g_0)g_0 - (1 + f_1)h_0,
\]
\[
(s_0 + f_0)h_j = -as_0g_j - ag_{j-1} - s_0\sum_{k=0}^{j} g_{j-k}g_k - \sum_{k=1}^{j} g_{j-k}g_{k-1}
- (1 + f_1)h_{j-1} - \sum_{k=0}^{j-2} f_{j-k}h_k, \quad j \geq 2.
\]

\[\textit{6.5. The } q,p \textit{ expansion about a regular point.} \] We will seek a Taylor series approximation for the scaled polynomial and associated function \( q(z; s), p(z; s) \) about the regular point \((z_0, s_0)\) with \(0 < z_0 < s_0\). Let us write

\[
q(z; s) = \sum_{j,k \geq 0} r_{j,k}(z - z_0)^j(s - s_0)^k,
\]
\[
p(z; s) = \sum_{j,k \geq 0} u_{j,k}(z - z_0)^j(s - s_0)^k,
\]

Using the first spectral derivative (5.34) we obtain the recurrence relation

\[
z_0(s_0 - z_0)(j + 1)r_{j+1,k} + (s_0 - 2z_0)jr_{j,k} - (j - 1)r_{j-1,k} + z_0(j + 1)r_{j+1,k-1} + jr_{j,k-1}
= -z_0u_{j,k} - u_{j-1,k} - \sum_{l=0}^{k} [f_{k-l}u_{j,l} + g_{k-l}r_{j-1,l} + z_0g_{k-l}r_{j,l}],
\]
for $j, k \geq 1$. When $k = 0$ and $j \geq 1$ we have the specialisation

\begin{equation}
 (6.59) \quad z_0(s_0 - z_0)(j + 1)r_{j+1,0} + [(s_0 - 2z_0)j + z_0g_0]r_{j,0} + [g_0 - j + 1]r_{j-1,0} = -(f_0 + z_0)u_{j,0} - u_{j-1,0}.
\end{equation}

The second spectral derivative (5.35) yields the recurrence relation

\begin{equation}
 (6.60) \quad z_0(s_0 - z_0)(j + 1)u_{j+1,k} + [(s_0 - 2z_0)j + 2s_0 - (a + 2)z_0]u_{j,k} - (a + j + 1)u_{j-1,k}
 + z_0(j + 1)u_{j+1,k-1} + (j + 2)u_{j,k-1}
 = \frac{1}{4}z_0(s_0 - z_0)r_{j,k} + \frac{1}{4}(s_0 - 2z_0)r_{j-1,k} - \frac{1}{4}r_{j-2,k} + \frac{1}{4}z_0r_{j,k-1} + \frac{1}{4}r_{j-1,k-1}
 - \sum_{l=0}^{k} h_{k-l}(r_{j-1,l} + z_0r_{j,l}) + \sum_{l=0}^{k} g_{k-l}(u_{j-1,l} + z_0u_{j,l}),
\end{equation}

for $j, k \geq 1$. When $k = 0$ and $j \geq 1$ we have the specialisation

\begin{equation}
 (6.61) \quad z_0(s_0 - z_0)(j+1)u_{j+1,0} + [(s_0 - 2z_0)j + 2s_0 - (a + 2)z_0 - z_0g_0]u_{j,0} - [a + 1 + g_0 + j]u_{j-1,0}
 = \left[\frac{1}{4}z_0(s_0 - z_0) - h_0z_0\right]r_{j,0} + \left[\frac{1}{4}(s_0 - 2z_0) - h_0\right]r_{j-1,0} - \frac{1}{4}r_{j-2,0}.
\end{equation}

Using the first deformation derivative (5.36) we obtain the recurrence relation

\begin{equation}
 (6.62) \quad s_0(s_0 - z_0)(k+1)r_{j,k+1} - s_0(k+1)r_{j-1,k+1} + (2s_0 - z_0)kr_{j,k} - kr_{j-1,k} + (k-1)r_{j,k-1}
 = s_0u_{j,k} + u_{j,k-1} + \sum_{l=0}^{k} \left[f_{k-l}u_{j,l} + g_{k-l}r_{j-1,l} + z_0g_{k-l}r_{j,l}\right],
\end{equation}

for $j, k \geq 1$. When $j = 0$ and $k \geq 1$ we have the specialisation

\begin{equation}
 (6.63) \quad s_0(s_0 - z_0)(k + 1)r_{0,k+1} + (2s_0 - z_0)kr_{0,k} + (k - 1)r_{0,k-1}
 = s_0u_{0,k} + u_{0,k-1} + \sum_{l=0}^{k} \left[f_{k-l}u_{0,l} + z_0g_{k-l}r_{0,l}\right].
\end{equation}

The second deformation derivative (5.37) in turn gives us the recurrence relation

\begin{equation}
 (6.64) \quad s_0(s_0 - z_0)(k + 1)u_{j,k+1} - s_0(k + 1)u_{j-1,k+1}
 + [(2s_0 - z_0)k + as]u_{j,k} - ku_{j-1,k} + (a + k - 1)u_{j,k-1}
 = \sum_{l=0}^{k} h_{k-l}(r_{j-1,l} + z_0r_{j,l}) - (2s_0 - z_0)\sum_{l=0}^{k} g_{k-l}u_{j,l} + \sum_{l=0}^{k} g_{k-l}r_{j-1,l} - 2\sum_{l=0}^{k} g_{k-l}u_{j,l},
\end{equation}
for $j,k \geq 1$. When $j = 0$ and $k \geq 1$ we have the specialisation

\begin{equation}
(6.65) \quad s_0(s_0 - z_0)(k + 1)u_{0,k+1} + [(2s_0 - z_0)k + a s_0]u_{0,k} + [a + k - 1]u_{0,k-1} = z_0 \sum_{l=0}^{k} h_{k-l}r_{0,l} - (2s_0 - z_0) \sum_{l=0}^{k} g_{k-l}u_{0,l} - 2 \sum_{l=0}^{k-1} g_{k-1-l}u_{0,l}.
\end{equation}

These recurrences can be solved in the following way. First (6.59) and (6.61) are solved for $r_{j,0}, u_{j,0}$ for $j \geq 1$ in terms of $r_{0,0}, u_{0,0}$. Then these solutions can be substituted into the boundary conditions

\begin{align}
(6.66) \quad \sum_{j \geq 0} r_{j,0}(-z_0)^j &= 1, \\
(6.67) \quad \sum_{j \geq 0} u_{j,0}(-z_0)^j &= 0,
\end{align}

and this allows for $r_{0,0}, u_{0,0}$ to be found. Next the sequence $r_{0,k}, u_{0,k}$ can be found for $k \geq 1$ using (6.63) and (6.65). Finally the two general systems (6.58),(6.60) and (6.62),(6.64) can be employed to compute $r_{j,k}, u_{j,k}$ for $j,k \geq 1$. 
6.6. Numerical studies at the Hard Edge. Using the integral formula for the
distribution $A_a(z)$ as given by (5.71) it is possible to compute values of this and
the examples of $a = 0, 1, 2$ are plotted in Figure 2. However we wish to characterise
it in a precise quantitative way and evaluate the moments of this distribution

\begin{equation}
(6.68) \quad m_k := \int_0^\infty dz \; z^k A_a(z), \quad k \in \mathbb{Z}_{\geq 0}.
\end{equation}

These are easily seen to be

\begin{equation}
(6.69) \quad m_k = \frac{1}{4^{2a+3}\Gamma(a+1)\Gamma(a+2)\Gamma^2(a+3)}
\times \int_0^\infty ds \; s^{k+2a+3}e^{-s/4}e^{F(s)} \int_0^1 du \; u^{k+2}(1-u)^a G(u; s),
\end{equation}

where

\begin{equation}
(6.70) \quad F(s) := \int_0^s \frac{dv}{v} \left[ \nu(v) + 2C(v) \right],
\end{equation}

and

\begin{equation}
(6.71) \quad G(z; s) := q\partial_z p - p\partial_z q.
\end{equation}

We note that by employing the large $s$ asymptotic form of $q(z; s), p(z; s)$ as given in
(6.39, 6.40) we can deduce the asymptotic form of the spacing distribution is given
by

\begin{equation}
(6.72) \quad A_a(z) \sim Ce^{-z/4+(a+2)\sqrt{z}}z^{-1/2-a^2/4},
\end{equation}

where $C$ is a constant which cannot be found from our methods. For small $z$ we
find that

\begin{equation}
(6.73) \quad A_a(z) \sim \frac{1}{4^{2a+3}\Gamma(a+1)\Gamma(a+2)\Gamma^2(a+3)}
\times \int_0^\infty ds \; s^{2a} \left( \frac{1}{12} - \frac{\xi(s)}{3s} \right) \exp \left( -\frac{s}{4} + \int_0^s \frac{dv}{v} \left[ \nu(v) + 2C(v) \right] \right) z^2,
\end{equation}

where we have used (5.58, 5.59) and the fact that $u_1^0(s) = 1/12 - \xi(s)/3s$. Therefore
we can conclude that the moments exist for $\text{Re}(k) > -3, \text{Re}(a) > -1$ and $\text{Re}(k + 2a) > -4$. An instance where exact evaluation of the moments can be made is the
case $a = 0$ and the first four of these are

\begin{equation}
(6.74) \quad m_1 = 4e^2 \left[ I_0(2) - I_1(2) \right], \quad m_2 = 32e^2 I_0(2),
\end{equation}

\begin{equation}
(6.75) \quad m_3 = 384e^2 \left[ 2I_0(2) + I_1(2) \right], \quad m_4 = 2048e^2 \left[ 13I_0(2) + 9I_1(2) \right].
\end{equation}

We investigated the distributions $A_a(z)$ and $A^\pm(z)$ for the two special cases of
$a = \pm 1/2$ in some detail because of the motivations provided by (1.6) and (1.7).
The analogue of (6.69) for \( A^\pm(z) \) is

\[
(6.76) \quad m_k^\pm = \frac{1}{2^{4a+5}\Gamma(a+1)\Gamma(a+2)\Gamma^2(a+3)} 
\times \int_0^\infty ds \, s^{k/2+2a+3}e^{-s/4}e^{F(s)} \int_0^1 du \, u^{k+2}(1-u)^{2a+1}(2-u)^2G(u(2-u)s;s),
\]

for \( a = \pm \frac{1}{2} \). The statistical data for \( A_{a}(z) \) for the cases \( a = -1/2, 0, 1/2, 1, 2 \) are given in Table 1 and the data for \( A^\pm(z) \) is given in Table 2.

Our strategy is that by employing local Puiseux-type and Taylor expansions for the two factors in the integrand, namely \( e^{F(s)} \) and \( G(z;s) \), within a given finite member of the patchwork of local expansion domains the above integrals restricted to this domain can be exactly evaluated. This is essential as numerical quadrature algorithms implemented in either computer algebra software or compiled language packages (e.g. QUADPACK) have minimal attained error tolerances which cannot be reduced below a fixed bound. For the compiled language option with a floating point representation of 64 bits the best one could expect is a relative error of around \( 10^{-15} \) but often it is far worse and around \( 10^{-8} - 10^{-9} \). To illustrate this we have computed the statistical data for the \( a = 1, 2 \) cases using QUADPACK routines and the results are displayed in Table 1. In the case of the Puiseux-type expansions the integrals are

\[
(6.77) \quad \int_0^S ds \, s^{k+3+l+u+(m+2)a}e^{-s/4},
\]

and

\[
(6.78) \quad \int_0^{1/2} du \, u^{k+2+n}(1-u)^a \quad \text{or} \quad \int_{1/2}^1 du \, u^{k+2}(1-u)^{a+n},
\]

which for \( k, l, m, n \in \mathbb{Z}_{\geq 0} \) can be evaluated in terms of radicals, the Gamma function at integer arguments, a rational function of \( a \) and the Whittaker function \( M(\alpha, \beta; S/4) \) or its specialisations depending on \( a \). For example in the case \( a = 1/2 \) the \( s \)-integral reduces to the error and exponential functions. For the Taylor expansion case we have the double integral

\[
(6.79) \quad \int_{s_1}^{s_2} ds \, s^{k+3+2a}(s-s_0)^m e^{-s/4} \int_0^1 du \, u^{k+2}(1-u)^a(us-z_0)^l,
\]

which for \( k, l, m \in \mathbb{Z}_{\geq 0} \) and \( s_0 \in (s_1, s_2) \) is evaluated in terms of a rational function of \( a \), a polynomial function of \( s_0, z_0 \) and the Whittaker functions \( M(\alpha, \beta; s_{1,2}/4) \). Therefore the only sources of error are from the truncation of the expansions and the finite number of intervals, both of which can be adjusted to reduce the contributing errors.

The computations were performed, in most cases, using the computer algebra system Maple with a sufficiently large number of decimal digits and found that 250 digits was more than adequate. In addition we found that we had to tailor
the numerical parameters for each case of \( a = \pm 1/2 \) differently as the errors varied quite strongly with \( a \) (this was especially pronounced as \( a \) approached \(-1\)). We discuss the case of \( a = 1/2 \) first. In regard to the truncations about the singular point \( s = 0 \), we found that 1890 terms in the expansion of the transcendent variable (6.1) with \( k \leq 30, j + k \leq 120 \) yielded an error for the second derivative of \( \nu(s) \) at \( s = 2 \) which was estimated to be \( 4.6 \times 10^{-38} \). At most 100 terms were retained in each of the expansions of the transcendent variables about regular points (6.41), (6.46), (6.47) and (6.48) because much fewer, of the order of \( j, k \leq 20 \), were required in the corresponding expansions of the linear variables. The number of intervals in the \( s \)-direction was taken to be 19 with the sequence of \( s_0 \) values being \( \{0, 2, 5/2, 3, 4, 6, 9, 13, 19, 25, 30, 38, 54, 72, 90, 115, 150, 200, 300, 500\} \). The boundaries of the \( s \)-interval with node \( s_0 \) were taken to be located at the midpoints of \( s_0 \) and its neighbouring nodes. This sequence of nodes was chosen to be close to an optimal situation yielding the largest separation of each node from its preceding node, yet close enough so to ensure that the error in \( \nu''(s) \) at the node was less than \( 9.8 \times 10^{-37} \). For each \( s \)-interval with node \( s_0 \) two expansion points in the \( z \)-direction were chosen because a single expansion point could never ensure that all of the integration region would fall within the domain of convergence about that point. The two points that together yielded the largest convergence domain were found to be located at \( z_0 = 0 \) and \( z_0 = s_0 \). Subdividing the \( z \)-interval into three sub-intervals was found to contribute a variation of less than \( 3.2 \times 10^{-19} \) to the normalisation. Another criteria that the sequence of \( s_0 \) nodes had to satisfy was that each \((s, z)\) integration region fell completely within the union of the convergence domains about \((s_0, 0)\) and \((s_0, s_0)\). For the expansions of the linear variables about the lines \( z = 0, s \) and about the singular point \( s = 0 \), as defined in (6.18, 6.19) along with (5.58, 5.59, 5.63, 5.64), we chose the cut-off in the sum to be 20. The expansions of linear variables about the lines \( z = 0, s \) and about a regular point \( s = s_0 > 0 \) as defined in (6.56, 6.57) were cutoff at 25. An overall estimate of the accuracy is provided by the normalisation, which was unity to within \( 1.6 \times 10^{-18} \). The second case with \( a = -1/2 \) was more demanding computationally. We needed 5150 terms in the expansion of the transcendent variable (6.1) with \( k \leq 50, j + k \leq 200 \) and computed these with compiled code using the multiple-precision library MPFUN [3],[2]. The error for \( \nu''(s) \) at \( s = 2 \) was estimated to be around \( 7.9 \times 10^{-59} \). Again only 100 terms were retained in each of the expansions of the transcendent variables about regular points. A larger number of intervals in the \( s \)-direction were employed, namely 24, and the sequence of \( s_0 \) values was taken to be \( \{0, 2, 5/2, 3, 7/2, 4, 5, 6, 7, 9, 11, 14, 18, 22, 28, 36, 46, 58, 72, 90, 114, 144, 180, 220, 300\} \). This time the sequence of nodes was chosen to ensure that the error in \( \nu''(s) \) at each node was less than \( 3.6 \times 10^{-60} \). And again for each \( s \)-interval with node \( s_0 \) two expansion points in the \( z \)-direction were chosen at \( z_0 = 0, s_0 \). In the expansions of
the linear variables about the lines \( z = 0, s \) and about the singular point \( s = 0 \) the cut-off in the sum was chosen to be 20 as before. The expansions of these variables about the lines \( z = 0, s \) and about a regular point \( s = s_0 > 0 \) was terminated at the cutoff of 25 also. The estimate of the accuracy provided by the normalisation was \( 4.6 \times 10^{-20} \).

Because the raw moments grow rapidly with order we have computed some standard statistical quantities instead using the definitions of the variance \( \sigma^2 \), the skewness \( \gamma_1 \) and the kurtosis excess \( \gamma_2 \)

\[
\sigma^2 = \mu_2, \quad \gamma_1 = \frac{\mu_3}{\mu_2^{3/2}}, \quad \gamma_2 = \frac{\mu_4}{\mu_2} - 3,
\]

in terms of the central moments

\[
\mu_2 = m_2 - m_1^2,
\]

\[
\mu_3 = m_3 - 3m_1m_2 + 2m_1^3,
\]

\[
\mu_4 = m_4 - 4m_1m_3 + 6m_1^2m_2 - 3m_1^4.
\]
Figure 2. The distribution of the first eigenvalue spacing at the hard edge of random hermitian matrices $A_a(z)$ for integral values of $a = 0, 1, 2$.

Acknowledgements

This work was supported by the Australian Research Council.
Table 1. Low order statistics of the distribution $A_a(z)$ for various values of the parameter $a$.

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References


Table 2. Low order statistics of the distribution $A_{\pm}^\pm(z)$ for the special cases of the parameter $a = \pm \frac{1}{2}$.


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