Exact Wigner surmise type evaluation of the spacing distribution in the bulk of the scaled random matrix ensembles

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Random matrix ensembles with orthogonal and unitary symmetry correspond to the cases of real symmetric and Hermitian random matrices respectively. We show that the probability density function for the corresponding spacings between consecutive eigenvalues can be written exactly in the Wigner surmise type form $a(s)e^{-b(s)}$ for $a$ simply related to a Painlevé transcendent and $b$ its anti-derivative. A formula consisting of the sum of two such terms is given for the symplectic case (Hermitian matrices with real quaternion elements).

PACS numbers: 05.45.+b, 05.40.+j

It is well established that universal features of the spectrum of classically chaotic quantum systems are correctly described by random matrix ensembles of an appropriate symmetry [1, 2, 3]. Generically there are three symmetry classes corresponding to two distinct time reversal symmetries plus the situation in which time reversal symmetry is absent. The level repulsion — which is a characteristic of the spectra of chaotic systems and is not present in the spectra of integrable systems — differs for each of the symmetry classes. Thus let $p(s)$ denote the probability density function for the spacing between consecutive levels, calculated after the energy levels are first rescaled so that the mean spacing is unity. Then $p(s) \propto s$ for a time reversal symmetry $T$ such that $T^2 = 1$, $p(s) \propto s^2$ in the absence of time reversal symmetry while $p(s) \propto s^4$ for a time reversal symmetry such that $T^2 = -1$. It is therefore convenient to distinguish the three cases by the label $\beta$ and so write $p_\beta(s)$, where $\beta = 1, 2$ or 4 depending on the small $s$ behaviour of $p_\beta(s)$. In this work succinct expressions for $p_\beta(s)$ will be given in terms of Painlevé transcendents.

The full probability density function $p_\beta(s)$ is by far the most studied statistic in relation to empirical data. For example in the early work [4] (this article is reprinted in [1]) on the energy levels of complex nuclei one sees in Figure 3 an empirical bar graph of $p_1(s)$ obtained from experimental data plotted on the same graph as a theoretical approximation to $p_1(s)$ known as the Wigner surmise. This approximation is given by the functional form $p_1^W(s) = (\pi s/2)e^{-(\pi s/2)^2}$. The fact that the Wigner surmise is an approximation rather than exact was soon realized [5], and the task of calculating the exact form of $p_1(s)$ was undertaken [6]. Actually
the quantity calculated was \( E_\beta(0; s) \), the probability there are no eigenvalues in an interval of length \( s \). One then computes \( p_\beta(s) \) using numerical differentiation via the formula

\[
p_\beta(s) = \frac{d^2}{ds^2} E_\beta(0; s).
\] (1)

The quantity \( E_1(0; s) \) was tabulated by first obtaining a formula involving the eigenfunctions and eigenvalues of a certain integral operator. The same general approach can be taken to compute \( p_\beta(s) \) for \( \beta = 2 \) and \( \beta = 4 \) [3]. The work [6] revealed that the Wigner surmise, although an excellent approximation, can be in error by up to 2% from the exact value.

There is a celebrated example of the empirical determination of \( p_2(s) \) which is so accurate that the exact value to an accuracy of three to four decimal places is essential. This occurs in Odlyzko’s numerical computation [7, 8] of the large zeros of the Riemann zeta function on the critical line, in particular zero number \( 10^{20} \) and \( 10^7 \) of its neighbours, which according to the Montgomery-Odlyzko law [9] are conjectured to have the same statistical properties as the eigenvalues of large dimensional random matrices.

Nearly 20 years after the numerical computation of \( E_1(0; s) \), Jimbo et al. [10] computed the exact functional form of \( E_2(0; s) \) in terms of a Painlevé V transcendent. Thus with \( \sigma(s) \) defined as the solution of the nonlinear equation

\[
(s\sigma'')^2 + 4(s\sigma' - \sigma)(s\sigma' - \sigma + (\sigma')^2) = 0
\]

subject to the boundary condition \( \sigma(s) \sim -s/\pi - (s/\pi)^2 \) as \( s \to 0 \), it was shown

\[
E_2(0; s) = \exp \left( \int_0^{\pi s} \frac{\sigma(t)}{t} \, dt \right).
\] (2)

Furthermore, using a known inter-relationship between \( E_2 \) and \( E_1 \) [11] a formula equivalent to

\[
E_1(0; s) = \left( E_2(0; s) \right)^{1/2} \exp \left( \frac{1}{2} \int_0^{\pi s} \left( -\frac{d}{dx} \frac{\sigma(x)}{x} \right)^{1/2} \, dx \right)
\] (3)

was presented. With \( E_2 \) and \( E_1 \) determined, \( E_4 \) can be computed from the formula [12]

\[
E_4(0; s/2) = \frac{1}{2} \left( E_1(0; s) + \frac{E_2(0; s)}{E_1(0; s)} \right).
\] (4)

Thus the exact functional form of \( p_\beta(s) \) can be obtained by substituting (2), (3) and (4) as appropriate in (1) and computing the second derivative. However the resulting formulas lack the aesthetic appeal of (2)–(4), and from a practical viewpoint have the drawback of requiring not only the computation of \( \sigma(x) \) but also its first and second derivative.

In this work we will show that for \( \beta = 1 \) and \( \beta = 2 \) the derivative

\[
\frac{d}{ds} E_\beta(0; s)
\]

can be written in a form analogous to (2), thus allowing expressions for \( p_\beta(s) \) to be obtained which have the Wigner surmise type structure \( p_\beta(s) = a(s)e^{-b(s)} \). A similar result will be obtained in the case \( \beta = 4 \). The starting point for our calculation in the case \( \beta = 2 \) is the
formula \((2)\), but in the case \(\beta = 1\) we use a formula distinct from \((3)\). This latter formula requires introducing a Painlevé transcendent \(\sigma_B\) (the use of the subscript \(B\) is motivated by the relation of this function to the Bessel kernel \([13]\)) which satisfies the nonlinear equation
\[
(s\sigma_B'')^2 + \sigma_B' (\sigma_B - s\sigma_B')(4\sigma_B' - 1) - \frac{1}{4}(\sigma_B')^2 = 0
\]  
subject to the boundary condition
\[
\sigma_B(s) \sim \frac{s^{1/2}}{\pi} + \frac{2s}{\pi^2}.
\]  
In terms of this function we have \([14]\)
\[
E_1(0; s) = \exp\left(- \int_0^{(\pi s/2)^2} \frac{\sigma_B(x)}{x} \, dx\right).
\]  
Our objective is to express the derivatives of \((2)\) and \((7)\) in terms of functional forms analogous to the original expressions. This is achieved by using some mathematical results \([15]\) relating to certain second order differential equations, of the second degree in \(y''\), which possess the Painlevé property. We find the functional forms involve different Painlevé transcendents to those occurring in \((2)\) and \((7)\). In the case \(\beta = 2\), the required Painlevé transcendent is specified by the solution of the nonlinear equation
\[
s^2(\hat{\sigma}'')^2 + 4(s\hat{\sigma}' - \hat{\sigma})(s\hat{\sigma}' - \hat{\sigma} + (\hat{\sigma}')^2) - 4(\hat{\sigma}')^2 = 0
\]  
subject to the boundary condition
\[
\hat{\sigma}(s) \sim -\frac{s^3}{3\pi}.
\]  
For \(\beta = 1\) the corresponding Painlevé transcendent is specified by the solution of the nonlinear equation
\[
s^2(\hat{\sigma}_B''')^2 = 4(\hat{\sigma}_B')^2 - \hat{\sigma}_B'(s\hat{\sigma}_B' - \hat{\sigma}_B) + \frac{9}{4}(\hat{\sigma}_B')^2 - \frac{3}{2}\hat{\sigma}_B + \frac{1}{4}
\]  
subject to the boundary condition
\[
\hat{\sigma}_B(s) \sim \frac{s}{3} - \frac{s^2}{45} + \frac{8s^{5/2}}{135\pi}.
\]  
In terms of the transcendents specified by \((8)\) and \((9)\) our results are
\[
\frac{d}{ds} \exp\left(\int_0^{\pi s} \frac{\sigma(t)}{t} \, dt\right) = -\exp\left(\int_0^{(\pi s/2)^2} \frac{\sigma_B(t)}{t} \, dt\right)
\]  
Recalling \((2)\), \((7)\) and \((1)\) we therefore have
\[
p_2(s) = -\frac{\hat{\sigma}(\pi s)}{s} \exp\left(\int_0^{\pi s} \frac{\hat{\sigma}(t)}{t} \, dt\right)
\]  
\[
p_1(s) = \frac{2\hat{\sigma}_B((\pi s/2)^2)}{s} \exp\left(\int_0^{(\pi s/2)^2} \frac{\hat{\sigma}_B(t)}{t} \, dt\right).
\]
In the case $\beta = 4$ we start with the formula [14]
\begin{equation}
E_4(0; s) = \frac{1}{2} \exp \left( - \int_0^{(\pi s)^2} \frac{\sigma_B(x)}{x} \, dx \right) + \frac{1}{2} \exp \left( - \int_0^{(\pi s)^2} \frac{\sigma_{B+}(x)}{x} \, dx \right)
\end{equation}
where $\sigma_{B+}$ satisfies the same d.e. (5) as $\sigma_B$, but is subject to the boundary condition
\begin{equation}
\sigma_{B+}(s) \sim \frac{s^{3/2}}{3\pi} \left( 1 + O(s) \right) + \frac{2}{3} \left( \frac{1}{3\pi} \right)^2 s^3 \left( 1 + O(s) \right).
\end{equation}
Proceeding as in the derivation of (12) we can derive the result
\begin{equation}
\frac{d}{dx} \exp \left( - \int_0^x \frac{\sigma_{B+}(t)}{t} \, dt \right) = -\frac{x^{1/2}}{3\pi} \exp \left( - \int_0^x \frac{\sigma_{B+}(t)}{t} \, dt \right)
\end{equation}
where $\sigma_{B+}$ satisfies the differential equation
\begin{equation}
\sigma''_{B+} = (4\sigma_{B+}^2 - \sigma_{B+})(s\sigma_{B+}' - \sigma_{B+}) + \frac{25}{4}(\sigma_{B+})^2 - \frac{5}{2}\sigma_{B+} + \frac{1}{4}
\end{equation}
subject to the boundary condition
\begin{equation}
\sigma_{B+}(s) \sim \frac{s}{5} \left( 1 + O(s) \right) + \frac{8s^{7/2}}{3^3 \cdot 5^3 \cdot 7\pi} \left( 1 + O(s) \right).
\end{equation}
We then have
\begin{equation}
p_4(s) = 2p_1(2s) + \frac{2\pi^2s}{3} \left( \sigma_{B+}((\pi s)^2) - 1 \right) \exp \left( - \int_0^{(\pi s)^2} \frac{\sigma_{B+}(t)}{t} \, dt \right).
\end{equation}

The formulas (13), (14) and (18) are our main results, giving exact functional forms of a Wigner surmise type structure for the universal spacing probabilities $p_1(s)$, $p_2(s)$ and $p_4(s) = 2p_1(2s)$. These expressions are well suited to the numerical tabulations of the $p_\beta(s)$, or the generation of power series expansions thereof, although that is not our concern here (accurate tabulations can be found in [16]).

Let us now turn to the derivation of (13), (14) and (18). Consider for example (14) (the derivation of (13) and (18) is similar; also the derivation of (13) will be presented as part of a forthcoming publication [17]). The key ingredient is the fact from [15] that the second order second degree equation
\begin{equation}
x^2(y'') = -4(y')^2(xy' - y) + A_2(xy' - y) + A_3y' + A_4
\end{equation}
is solved in terms of a particular Painlevé V transcendent $u$ satisfying
\begin{equation}
u'' = \left( \frac{1}{2u} + \frac{1}{u-1} \right) (u')^2 - \frac{u'}{x} + \frac{(u - 1)^2}{x^2} \left( \alpha u + \beta \right) + \frac{\gamma u}{x}.
\end{equation}
This is the Painlevé V equation with $\delta = 0$ in standard notation. As an aside we remark that it is known [15] that (20) can always be solved in terms of a Painlevé III transcendent. The relationship between (19) and (20) is via the formulas
\begin{align}
y &= \frac{1}{4u} \left( \frac{ux' - u}{u - 1} \right)^2 - \frac{1}{4} \left( 1 - \sqrt{2}\alpha \right)^2 (u - 1) - \frac{\beta u - 1}{2} \frac{u}{u} + \frac{\gamma x + 1}{4} \frac{u}{u - 1} \\
y' &= -\frac{x}{4u(u - 1)} \left( u' - \sqrt{2}\alpha \frac{u(u - 1)}{x} \right)^2 - \frac{\beta u - 1}{2x} \frac{u}{u} - \frac{\gamma}{4} \frac{u}{4} \\
A_2 &= \frac{\gamma^2}{4}, \quad A_3 = \gamma \left( \beta + \frac{1}{2} \left( 1 - \sqrt{2}\alpha \right)^2 \right), \quad A_4 = \frac{\gamma^2}{4} \left( -\beta + \frac{1}{2} \left( 1 - \sqrt{2}\alpha \right)^2 \right).
\end{align}
Now it is easy to see that (19) reduces to (5) if we write
\[ x \mapsto s, \quad y \mapsto -(\sigma_B - \frac{1}{8}s - \frac{1}{16}), \]  
(24)

\[ A_2 = \frac{1}{16}, \quad A_3 = -\frac{1}{16}, \quad A_4 = \frac{1}{128}. \]  
(25)

Substituting (25) in (23) gives
\[ \sqrt{2\alpha} = 1, \quad \beta = -\frac{1}{8}, \quad \gamma = \frac{1}{2}, \]  
(26)

while use of (24) and (26) in (21) and (22) allows us to deduce
\[ \frac{\sigma_B'}{\sigma_B} = -\frac{u - 1}{s}, \]  
(27)

Furthermore we observe from (21), (22) with the substitutions (24) and (25) that
\[ \sigma_B + (u - 1) + \frac{1}{2} =: \tilde{\sigma}_B \]  
(28)

is also of the form (21), (22) but with
\[ x \mapsto s, \quad y \mapsto -(\tilde{\sigma}_B - \frac{1}{8}s - \frac{9}{16}), \quad \sqrt{2\alpha} = -1, \quad \beta = -\frac{1}{8}, \quad \gamma = \frac{1}{2}. \]  
(29)

These last three values substituted in (25) gives \( A_2 = \frac{1}{16}, \ A_3 = \frac{15}{16}, \ A_4 = \frac{17}{128}, \) and these values together with the first two identifications in (29) substituted in (19) give the differential equation (9). The equation (12) follows from (27) and (28), together with the facts deducible from (6) that \( \tilde{\sigma}_B \sim 0 \) as \( s \to 0, \) while \( \pi\sigma_B(s)/s^{1/2} \sim 1 \) and the boundary condition (10) follows similarly.

We remark that the same procedure starting with \( \tilde{\sigma}_B \) instead of \( \sigma_B \) does not lead to a simple formula analogous to (27), so we cannot expect the derivative of the RHS of (12) to simplify; the results (13), (14) and (18) appear to be the simplest functional forms possible. We emphasize that the structure of these exact functional forms for \( p_1(s), \ p_2(s) \) and \( p_4(s) - 2p_1(2s) \) are of the Wigner surmise type \( a(s)e^{-b(s)}, \) where instead of \( a(s) \) and \( b(s) \) being simple power functions as in the approximation of Wigner, \( a(s) \) and \( b(s) \) are Painlevé transcendents.

**Acknowledgements**

This work was supported by the Australian Research Council.

**References**


